DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM – ANALYSIS
31 AUGUST 2000

Instructions: The exam consists of eight problems, all counted equally. You are encouraged to try to solve every problem; there is no penalty for incorrect answers. In order to pass the exam, it is enough to solve essentially correctly at least five problems and to have an overall score of at least 65%. State explicitly all results that you use in your proofs and verify that these results apply.

Conventions. 1_A denotes the indicator function of a set A. If the measure is not specified, use Lebesgue measure on $I\!R$. This measure is denoted by m.

1. (a) Let $\{f_n, n \in I\!N\}$ be a sequence of Lebesgue-measurable functions on $I\!R$ and let f be a Lebesgue-measurable function on $I\!R$. Assume that $f_n \geq 0$ and that $f_n \to f$ in measure. Prove that

$$\int_{I\!R} f\, dm \leq \liminf_{n\to\infty} \int_{I\!R} f_n\, dm.$$

(**Hint.** Consider a proof by contradiction.).

(b) Let E be a Lebesgue-measurable subset of $[0,1]$ satisfying $1 > m(E) > 0$. Define a sequence of functions $\{f_n, n \in I\!N\}$ as follows:

$$f_n = \begin{cases} 1_E & \text{if } n \text{ is odd} \\ 1 - 1_E & \text{if } n \text{ is even.} \end{cases}$$

Compute $f = \liminf_{n\to\infty} f_n$ and show that

$$\int_{[0,1]} f\, dm < \liminf_{n\to\infty} \int_{[0,1]} f_n\, dm.$$

2. Let $\{f_n, n \in I\!N\}$ be a sequence of Lebesgue-measurable functions on $[0,1]$ for which there exist real constants $c < \infty$ and $p > 1$ with the following property:

$$\int_{[0,1]} |f_n|\, dm \leq \frac{c}{n^p} \quad \text{for all } n.$$

Prove that $f_n \to 0$ a.e. (**Hint.** Consider $g_n = \sum_{k=1}^n |f_k|$.)

3. Let f be a function in $L^1(\mathbb{R})$. For $t \in \mathbb{R}$ define

$$\psi(t) = \int_{\mathbb{R}} \cos(tx)\, f(x)\, dm(x).$$

(a) Prove that $\psi(t)$ is a continuous function of $t \in \mathbb{R}$.

(b) Assume that $\int_{\mathbb{R}} |xf(x)|\, dm(x) < \infty$. Prove that $d\psi(t)/dt$ exists and equals

$$\int_{\mathbb{R}} \frac{d\cos(tx)}{dt} f(x)\, dm(x) = -\int_{\mathbb{R}} x \sin(tx)\, f(x)\, dm(x).$$

4. Let μ be a finite measure on a measurable space $(\mathcal{X}, \mathcal{M})$.

(a) Let $\{E_j, j \in \mathbb{N}\}$ be a sequence of disjoint subsets of $\mathcal{M}$ and define $E = \cup_{j \in \mathbb{N}} E_j$. Prove that as $n \to \infty$ $\sum_{j=1}^n 1_{E_j} \to 1_E$ in $L^1(\mu)$.

(b) Let φ be a bounded linear functional mapping $L^1(\mu)$ into $[0, \infty)$. Using part (a), prove that the function mapping $A \in \mathcal{M} \mapsto \varphi(1_A)$ is a finite measure on $(\mathcal{X}, \mathcal{M})$ and that this measure is absolutely continuous with respect to μ.

(c) As in part (b), let φ be a bounded linear functional mapping $L^1(\mu)$ into $[0, \infty)$. What theorem allows you to conclude from part (b) that there exists a nonnegative function $g \in L^1(\mu)$ such that $\varphi(1_A) = \int_A g\, d\mu$ for all $A \in \mathcal{M}$? Justify your answer.

5. Let $\mathcal{H}$ be a Hilbert space with norm $\|\cdot\|$ and let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators mapping $\mathcal{H}$ into $\mathcal{H}$. For $A \in \mathcal{B}(\mathcal{H})$ define the operator norm

$$\|A\| = \sup\{\|Ax\| : x \in \mathcal{H}, \|x\| = 1\} = \sup\left\{\frac{\|Ax\|}{\|x\|} : x \in \mathcal{H}, x \neq 0\right\}.$$

(a) For A and B in $\mathcal{B}(\mathcal{H})$ and $x \in \mathcal{H}$, define $(AB)x = A(Bx)$. Prove that $\|AB\| \leq \|A\| \cdot \|B\|$.

(b) Prove that every Cauchy sequence $\{A_n, n \in \mathbb{N}\}$ in $\mathcal{B}(\mathcal{H})$ converges in $\mathcal{B}(\mathcal{H})$.

6. Let α be a number in $(1, \infty)$. Define the measure λ on Borel subsets A of $[0, \infty)$ by

$$\lambda(A) = \int_A (1 + x^\alpha)^{-1} \, dm(x).$$

(a) Prove that λ is absolutely continuous with respect to m and determine $d\lambda/dm$, the Radon-Nikodym derivative of λ with respect to m.

(b) For which values of $p \in \mathbb{R}$ is $f(x) = x^p$ in $L^1(\lambda)$? Justify your answer.

(c) For which values of $p \in \mathbb{R}$ is $f(x) = x^p$ in $L^2(\lambda)$? Justify your answer.

7. Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let $\{v_1, v_2, \ldots, v_N\}$ be a finite orthonormal set in $\mathcal{H}$. Let $y = \sum_{k=1}^N c_k v_k$.

(a) Prove the following: (i) $c_k = \langle y, v_k \rangle$; (ii) $\|y\|^2 = \sum_{k=1}^N |c_k|^2$.

(b) Define M to be the closed subspace of $\mathcal{H}$ spanned by $\{v_1, v_2, \ldots, v_N\}$. A standard fact about Hilbert spaces (which you need not prove) is the following: for any $x \in \mathcal{H}$ there exists a unique element $x_0 \in M$ which minimizes the distance $\{\|x - z\| : z \in M\}$ and also has the property that $x - x_0 \in M^\perp$. Prove that $x_0 = \sum_{k=1}^N \langle x, v_k \rangle v_k$.

(c) State and prove Bessel's inequality for $x \in \mathcal{H}$ relative to the orthonormal set $\{v_1, v_2, \ldots, v_N\}$.

8. Let f be a continuous function mapping $\mathbb{R}$ into $\mathbb{R}$ which is periodic of period 1 (thus $f(x+1) = f(x)$ for all $x \in \mathbb{R}$). Let α be an irrational real number. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^N f(k\alpha) = \int_0^1 f \, dm.$$

(**Hint.** First prove the limit for $f(x) = e^{2\pi i k x}, k \in \mathbb{Z}$).