UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Wednesday, August 25, 2021

Show all work in your solution to each problem. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. All five questions are worth 20 points each.

1. Let A, B, and E be any three events. Consider the following three statements. If false in general, give a proof or a simple, concrete counterexample. If true, give a proof.
   (a) $P(A \cup B) = P(A) + P(B) - P(A)P(B)$
   (b) $P(A \cap B) \geq P(A) + P(B) - 1$
   (c) If $P(A|E) \geq P(B|E)$ and $P(A|E^c) \geq P(B|E^c)$ then $P(A) \geq P(B)$.

2. A weed is exposed to a known dose of weed killer x. The weed either survives ($Y = 1$) or dies ($Y = 0$). Suppose the weed has an unobserved natural tolerance to the weed killer (denoted by Z), and assume that this tolerance has a standard normal distribution. Further, suppose that the weed dies if an only if $Z < x$. Note that Z is random and x is fixed. You may leave your responses to this question in terms of the standard normal PDF ϕ and CDF Φ.
   (a) What is the probability that the weed survives?
   (b) What is the density of Z given that the weed is not killed?
   (c) What is the derivative of the standard normal density function?
   (d) What is the conditional expectation of Z given that the weed survives?

3. Joe walks to and from work each day. The commute to work on day i, T_i, has mean μ_T and variance σ_T^2. The commute from work, F_i, has mean μ_F and variance σ_F^2. His commutes from day to day are independent of each other, i.e., T_i and F_i are independent of T_j and F_j when $i \neq j$. However, when Joe walks fast in the morning, he tends to work slower on his walk back, and vice versa: The covariance of D_i and F_i is γ. Let $D_i = T_i - F_i$.
   (a) What are the mean and variance of D_i?
   (b) Let $\overline{D}_{100}$ be the mean difference over 100 days: $\overline{D}_{100} = \sum_{j=1}^{100} D_i/100$. Write an approximation for the probability that $\overline{D}$ is negative.

4. Let X and Y have the joint density

$$f(x,y) = \frac{6}{7}(x+y)^2, \qquad 0 \leq x \leq 1, 0 \leq y \leq 1.$$

(a) Prove that $f(x, y)$ is a density.

(b) By integrating over the appropriate region, find $P(X > Y)$.

(c) Find the marginal densities of X and Y.

(d) Find the conditional density of X given Y.

5. Let X and Y be independent and identically distributed exponential(11 random variables, with densities $f(z) = exp(-z), z \geq 0$. You may use without proof that $X + Y$ has a gamma(2,1) distribution with density $f(z) = zexp(-z), z \geq 0$.

(a) You do not need to prove this, but how would you show that $X + Y$ has a gamma(2,1) distribution?

(b) Let $k > 0$. Show that $[X|X + Y = k]$ has a uniform(0,k) distribution.

(c) Show that $\left[\frac{X}{X+Y}|X + Y = k\right]$ is uniform(0,1).

(d) Why does that meant that $\frac{X}{X+Y}$ and $X + Y$ are independent?

(e) Why does that meant that $\frac{X+Y}{X}$ and $X + Y$ are independent?

(f) Why does that meant that $\frac{X}{Y}$ and $X + Y$ are independent?