DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
January 2013

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let Y_1 and Y_2 have the joint probability density function given by

$$f(y_1, y_2) = c(2 - y_1) \text{ for } 0 \le y_2 \le y_1 \le 2.$$

For the following parts, you can leave your answers in terms of integrals with explicit limits. No need to give the final numerical answers.

   (a) (6 points) Find c.

   (b) (6 points) Find the marginal density functions for Y_1 and Y_2.

   (c) (6 points) Are Y_1 and Y_2 independent? Why or why not?

   (d) (6 points) Find the conditional density of Y_1 given $Y_2 = y_2$ and $P(Y_1 \ge 1.5 | Y_2 = 0.9)$.

2. Let $X \sim N(\mu, \sigma^2)$.

   (a) (6 points) Show that the moment generating function of X is

$$M_X(t) = \exp(\mu t + \sigma^2 t^2/2).$$

   (b) (7 points) Show that if $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$, $X_3 \sim N(\mu_3, \sigma_3^2)$, and X_1, X_2, X_3 are independent, then $X_1 + X_2 + X_3 \sim N(\mu_1 + \mu_2 + \mu_3, \sigma_1^2 + \sigma_2^2 + \sigma_3^2)$.

   (c) (7 points) Suppose μ_i and σ_i, $i = 1, 2, 3$ are known quantities. Based on X_1, X_2, X_3, construct a statistic that has a Chi squared distribution with 3 degrees of freedom, a statistics that has a t distribution with 2 degrees of freedom, and a statistic that has an F distribution with 1 and 2 degrees of freedom.

   (d) (7 points) Suppose that $\mu_i = \mu$ and $\sigma_i^2 = \sigma^2$ for $i = 1, 2, 3$. Define $S^2 = \frac{1}{2}\sum_{i=1}^{3}(X_i - \bar{X})^2$ with $\bar{X} = (X_1 + X_2 + X_3)/3$. Show that $ES^2 = \sigma^2$ and $ES \le \sigma$.

3. Suppose (X_1, X_2, X_3) follows a multinomial distribution with m trials and cell probabilities $p_1, p_2, \cdots, p_3$. Note that the Multinomial(m, p_1, p_2, p_3) probability mass function is $\frac{m!}{x_1! \cdots x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}, \quad x_1 + x_2 + x_3 = m, \quad p_1 + p_2 + p_3 = 1$.

   (a) (7 points) Find the marginal distribution of X_1 and the conditional distribution of X_2 given X_1. Write down your reasoning. No mathematical proof is needed.

   (b) (7 points) Show that $Cov(X_1, X_2) = -m p_1 p_2$.

   (c) (7 points) Let $\begin{pmatrix} X_{11} \\ X_{21} \\ X_{31} \end{pmatrix}, \begin{pmatrix} X_{12} \\ X_{22} \\ X_{32} \end{pmatrix}, \cdots, \begin{pmatrix} X_{1n} \\ X_{2n} \\ X_{3n} \end{pmatrix}$ be iid random vectors from the above multinomial distribution. Let $\bar{X}_j = \frac{1}{n}\sum_{i=1}^{n} X_{ji}, j = 1, 2$. Find the limiting joint distribution of $(\bar{X}_1, \bar{X}_2)$. State the theorem used.

   (d) (7 points) Find the limiting distribution of $Y_n = \bar{X}_1 / \bar{X}_2$. State the theorem used.

4. Suppose we have three cards. The first one is blank on both sides, the second has an X on one side and is blank on the other, and the third has an X on both sides. We run an "experiment" where we choose one card at random and then look at one side of the chosen card at random.

(a) (7 points) What is the probability that you see an X?

(b) (7 points) What is the probability that you see an X and the other side of the chosen card has an X too?

(c) (7 points) Suppose we run the experiment above and we see an X. Given that outcome, what is the probability that the other side of the card has an X on it too?