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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers <u>clearly</u> in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra $\mathcal{B}(\mathbb{R})$

1. Let $(X, \mathcal{M})$ be a measurable space and let $f : X \to I\!R$ be a Borel-measurable function. Define $\mathcal{N}(f)$ to be the class of all sets of the form $f^{-1}(B)$ for $B \in \mathcal{B}(I\!R)$.

   (a) Prove that $\mathcal{N}(f)$ is a σ-algebra.

   (b) Let $\varphi : I\!R \to I\!R$ be a Borel measurable function. Prove that the function $\varphi \circ f$ is $\mathcal{N}(f)$ measurable.

   (c) Let $g : X \to I\!R$ be a simple function of the form $\sum_{i=1}^{s} c_i 1_{A_i}$, where c_i are real numbers and A_i are pairwise disjoint sets in $\mathcal{N}(f)$. Prove that there exists a Borel measurable function $\varphi : I\!R \to I\!R$ such that $g = \varphi \circ f$.

2. For $x \in [0,1]$, let

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}\,, \quad a_n \in \{0,1\}\,,$$

be the binary expansion of x. Let A be the set of points x which admit a binary expansion with zero in all even positions (i.e., $a_{2n} = 0$ for all $n \geq 1$). Show that A is a set of Lebesgue measure 0.

Hint: Write the set A has $A = \cap_{n=0}^{\infty} A_n$ where $A_0 = [0,1]$, the A_n are nested, i.e. $A_{n+1} \subset A_n$ and A_{n+1} is obtained from A_n by removing some of the dyadic intervals $(\frac{j}{2^n}, \frac{j+1}{2^n})$, $0 \leq j \leq 2^n - 1$ in A_n.

3. Let $f_n : (a, b) \to \mathbb{R}, n \geq 1$ be a sequence of functions each of which is a monotonically increasing function. Suppose the sequence f_n converges to f in Lebesgue measure on (a, b).

   Show that f_n converges to f at all points in (a,b) where f is continuous.

4. Let $A \subset \mathbb{R}$ and $f_n : A \to \mathbb{R}$, for $n \geq 1$. The family of function $\{f_n\}_{n\geq 1}$ is said to be *equi-integrable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $m(B) \leq \delta$ implies that $\int_B |f_n| dm \leq \epsilon$ for all $n \geq 1$.

   Let A be a set of finite measure, $m(A) < \infty$, and let $\{f_n\}_{n\geq 1}$ be a sequence of functions such that

   (i) $\{f_n\}_{n\geq 1}$ is equi-integrable.

   (ii) f_n converges pointwise to f Lebesgue almost everywhere in A.

   Show that
$$\lim_{n\to\infty} \int_A f_n \, dm = \int f dm \, .$$

5. Let us define the Haar functions on the interval $[0,1]$ by

$$e_0(x) = 1\,,$$

$$e_{n,k}(x) = \begin{cases} 2^{\frac{n}{2}} & \text{if } \frac{k-1}{2^n} \le x < \frac{k-\frac{1}{2}}{2^n} \\ -2^{\frac{n}{2}} & \text{if } \frac{k-\frac{1}{2}}{2^n} \le x < \frac{k}{2^n} \\ 0 & \text{otherwise}\,, \end{cases}$$

for $n = 0, 1, 2, 3, \cdots$ and $k = 1, 2, \cdots 2^n$. Show that the Haar functions form an orthornormal basis of the Hilbert space $L^2([0,1])$.

6. Let f be a Lebesgue integrable function on $(0, a)$, $a > 0$, and define

$$g(x) = \int_x^a \frac{f(t)}{t} dt, \quad 0 < x < a.$$

(a) Show that g is Lebesgue integrable on $(0, a)$.

(b) Show that

$$\int_0^a g(x)dx = \int_0^a f(t)dt.$$

Justify all your work!

7. (a) Show that $f(x) = x^2 \cos^2(\pi/x)$ is a function of bounded variation on $[0, 1]$.

   Hint: You may use some well-known theorem about function of bounded variation.

   (b) Show that the function $g(y) = \sqrt{y}$ is a function of bounded variation on $[0, 1]$.

   Hint: You may use some well-known theorem about function of bounded variation.

   (c) Is the composition of functions of bounded variation always of bounded variation?

   Hint: Consider $h = g \circ f$ with f, g as in (a),(b) and use the definition of bounded variation.

   (d) Show that if g satisfy a Lipschitz condition (i.e. there exists $L > 0$ such that $|g(x) - g(y)| \leq L|x - y|$ for all x, y) and f is of bounded variation then $h = g \circ f$ is of bounded variation.

8. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $0 < p < 1$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that if f and g are positive functions then

$$\int fg d\mu \geq \left(\int f^p d\mu \right)^{1/p} \left(\int g^q d\mu \right)^{1/q}.$$

Hint: Use Hölder inequality for suitable functions u and v.