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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.

2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.

3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and f a measurable function mapping X into $\mathbb{R}$. For each $n \in \mathbb{N}$ define

$$E_n = \{x \in X : (n-1) \leq |f(x)| < n\}.$$

(a) Prove that $f \in L^1(\mu)$ if and only if $\sum_{n=1}^{\infty} n\mu(E_n) < \infty$.

(b) Prove that $f \in L^p(\mu)$ for $1 < p < \infty$ if and only if $\sum_{n=1}^{\infty} n^p\mu(E_n) < \infty$.

2. Let g be a nonmeasurable function mapping $\mathbb{R}$ into $\mathbb{R}$. Define the function f mapping $\mathbb{R}^2$ into $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} g(x) & \text{for } x \in \mathbb{R} \text{ and } y \text{ rational} \\ \exp(-|x| - |y|) & \text{for } x \in \mathbb{R} \text{ and } y \text{ irrational.} \end{cases}$$

(a) Prove that f is measurable. (**Hint.** You may use without proof the fact that $m \times m(\mathbb{R} \times \{c\}) = 0$ for any $c \in \mathbb{R}$.)

(b) Prove that f is integrable and evaluate $\displaystyle\int_{\mathbb{R}^2} f(x,y)\,dx\,dy$.

3. Let $K(\cdot,\cdot)$ be in the real Hilbert space $L^2(\mathbb{R}\times\mathbb{R})$; i.e., $K(\cdot,\cdot)$ maps $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$ and

$$\|K\|_{L^2(\mathbb{R}\times\mathbb{R})} = \left(\int_{\mathbb{R}\times\mathbb{R}} |K(x,y)|^2\,dx\,dy\right)^{1/2} < \infty.$$

For f in the real Hilbert space $L^2(\mathbb{R})$ and $x \in \mathbb{R}$ define

$$Tf(x) = \int_{\mathbb{R}} K(x,y)\,f(y)\,dy.$$

Denote the norm in $L^2(\mathbb{R})$ by $\|\cdot\|$ and the inner product in that space by $\langle\cdot,\cdot\rangle$.

(a) Prove that if $f \in L^2(\mathbb{R})$, then $Tf \in L^2(\mathbb{R})$.

(b) Prove that

$$\sup\{\|Tf\| : f \in L^2(\mathbb{R}), \|f\| = 1\} \le \|K\|_{L^2(\mathbb{R}\times\mathbb{R})}.$$

(c) The adjoint T^* of T is uniquely determined by the formula $\langle Tf, g\rangle = \langle f, T^*g\rangle$ for all $f, g \in L^2(\mathbb{R})$. Using this formula, prove that for $g \in L^2(\mathbb{R})$ and $x \in \mathbb{R}$

$$T^*g(x) = \int_{\mathbb{R}} K(y,x)\,g(y)\,dy.$$

4. Let $(X, \mathcal{M}, \mu)$ be a measure space, A a set in $\mathcal{M}$, and f a measurable function mapping X into $\mathbb{R}$.

(a) Prove that if $f \geq 0$ and $\displaystyle\int_A f\, d\mu = 0$, then $f(x) = 0$ for μ–almost every $x \in A$.

(b) Prove that if $\displaystyle\int_B f\, d\mu = 0$ for all measurable subsets $B \subset A$, then $f(x) = 0$ for μ–almost every $x \in A$.

(c) Prove that if $\displaystyle\left|\int_A f\, d\mu\right| = \int_A |f|\, d\mu$, then either $f(x) \geq 0$ for μ–almost every $x \in A$ or $f(x) \leq 0$ for μ–almost every $x \in A$.

5. Fix real numbers a and b satisfying $-\infty < a < b < \infty$. Let $(X, \mathcal{M}, \mu)$ be a measure space and f a function mapping $X \times [a, b]$ into $\mathbb{R}$ with the following properties.

   (i) For all $x \in X$ and $t \in [a, b]$, $\dfrac{\partial f}{\partial t}(x, t)$ exists.

   (ii) There exists $g \in L^1(\mu)$ such that for all $x \in X$ and $t \in [a, b]$, $\left|\dfrac{\partial f}{\partial t}(x, t)\right| \leq g(x)$.

   For $t \in [a, b]$ define

$$F(t) = \int_X f(x, t)\, d\mu(x).$$

   Prove that $F(t)$ is differentiable for all $t \in [a, b]$ and that

$$F'(t) = \int_X \frac{\partial f}{\partial t}(x, t)\, d\mu(x).$$

   (**Hint.** Use the mean value theorem and a well known limit theorem.)

6. Let $(X, \mathcal{M})$ be a measurable space and μ, ν, and λ σ-finite, positive measures on $(X, \mathcal{M})$.

(a) Prove that $\mu \ll \mu + \nu$.

(b) Assume that $\nu \ll \mu$ and $\lambda \ll \mu$. Prove that $\nu + \lambda \ll \mu$ and

$$\frac{d(\nu + \lambda)}{d\mu} = \frac{d\nu}{d\mu} + \frac{d\lambda}{d\mu} \quad \mu\text{–a.e.}$$

(c) Assume that $\lambda \ll \nu$ and $\nu \ll \mu$. Prove that $\lambda \ll \mu$ and

$$\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu} \quad \mu\text{–a.e.}$$

7. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\{f_n, n \in \mathbb{N}\}$ a sequence of measurable functions mapping X into $\mathbb{R}$, and f a measurable function mapping X into $\mathbb{R}$.

(a) Define the concept that $f_n \to f$ in measure.

(b) Assume that $f_n \to f$ in measure. Prove that for any bounded, uniformly continuous function $h : \mathbb{R} \to \mathbb{R}$

$$\lim_{n\to\infty} \int_X h \circ f_n \, d\mu = \int h \circ f \, d\mu.$$

8. (a) Prove the following two algebraic identities.

(i) For $t \geq 0$, $t \leq \log(1 + e^t) \leq t + \log 2$.

(ii) For $t \leq 0$, $0 \leq \log(1 + e^t) \leq \log 2$.

(b) Let g be a Lebesgue integrable function mapping $[0, 1]$ into $\mathbb{R}$. Using part (a), compute the limit

$$\lim_{n \to \infty} \int_0^1 \frac{1}{n} \log(1 + e^{ng(x)})\, dx.$$

(**Hint.** Use the Dominated Convergence Theorem.)