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Advanced Analysis Qualifying Examination
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Wednesday, August 28th, 2019

Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all the results that you use in your proofs and verify that these results apply.
5. Show all your work and justify the steps in your proofs.
6. Please write your full work and answers clearly in the blank space under each question and on the blank page after each question.

Conventions

1. For a set A, χ_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}^d$. This measure is denoted by m or $m_{\mathbb{R}^d}$.
3. If a σ-algebra on $\mathbb{R}^d$ is not specified, use the Borel σ-algebra.

1. For $\alpha \in \mathbb{R}$, define the function $f_\alpha : \mathbb{R}^d \to \mathbb{R}$ by

$$f_\alpha = \sum_{n=1}^{\infty} n^\alpha \chi_{A_n}, \quad \text{where} \quad A_n = \left\{ x \in \mathbb{R}^d \ : \ \frac{1}{n+1} < |x| \le \frac{1}{n} \right\}.$$

Find the values of α for which f_α is integrable, and then prove that f_α is integrable for those values of α.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \mathcal{N})$ a measurable space. Let $h\colon X \to Y$ be a measurable function.

(a) Define a set function ν on $\mathcal{N}$ by $\nu(A) = \mu(h^{-1}(A))$ for every $A \in \mathcal{N}$. Prove that ν is a measure on $\mathcal{N}$.

(b) Prove that if $f \in L^1(\nu)$, then $f \circ h \in L^1(\mu)$ and that

$$\int_Y f \, d\nu = \int_X (f \circ h) \, d\mu.$$

(c) Consider $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N})$ as in line 1 of this problem and consider the measure ν defined in part (a). Assume that $\mu(X) < \infty$ and that γ is a finite measure on $(Y, \mathcal{N})$ satisfying $\nu \ll \gamma$. By using part (b) and quoting a well known theorem in measure theory, prove that there exists $g \in L^1(\gamma)$ such that

$$\int_X (f \circ h) \, d\mu = \int_Y f \, g \, d\gamma \text{ for each } f \in L^1(\nu).$$

3. (a) Define convergence in measure.

(b) Assume $m(E) < \infty$, and define

$$\rho(g, h) = \int_E \frac{|g - h|}{1 + |g - h|},$$

for any measurable functions g and h defined on E. Show that a sequence f_n converges in measure to f <u>if and only if</u>

$$\lim_n \rho(f_n, f) = 0.$$

4. Consider the function $f(x, y) := e^{-xy} - 2e^{-2xy}$ where $x \in (1, \infty)$ and $y \in (0, 1)$.

   (a) Prove that for a.e. $y \in (0, 1)$ $\;f^y\;$ (defined as $f^y(x) = f(x, y)$) is integrable on $(1, \infty)$ with respect to $m_{\mathbb{R}}$.

   (b) Prove that for a.e. $x \in (1, \infty)$ $\;f^x\;$ (defined as $f^x(y) = f(x, y)$) is integrable on $(0, 1)$ with respect to $m_{\mathbb{R}}$.

   (c) Use Fubini to prove that $f(x, y)$ is not integrable on $(1, \infty) \times (0, 1)$ with respect to $m_{\mathbb{R}^2}$.

5. (a) Let f be defined by $f(x) = x^2 \sin(1/x^2)$ for $x \neq 0$, and $f(0) = 0$. Does f have finite variation over the interval $[-1, 1]$? Justify your answer and show your work.

(b) Compute the Lebesgue-Stieljes integral

$$\int_{[-2,2]} x^2 \, dF(x), \quad \text{where} \quad F(x) = \begin{cases} x+1 & \text{if } -2 \leq x < -1, \\ 2 & \text{if } -1 \leq x < 0, \\ x^2+1 & \text{if } 0 \leq x < 2. \end{cases}$$

6. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let u and v be linearly independent, unit vectors in $\mathcal{H}$. Define M to be the linear span of u and v.

   (a) Determine a unit vector w such that $\langle u, w \rangle = 0$ and the linear span of u and w equals M. Be sure that you verify the latter statement about the linear span of u and w.

   (b) Let x be an element in $\mathcal{H} \setminus M$. Determine explicitly, in terms of u and w, a $y_0 \in M$ such that

$$\|x - y_0\| = \inf\{\|x - z\| : z \in M\}.$$

   (c) Prove that the y_0 found in part (b) is unique and re-express it in terms of u and v.

7. Let X be a Banach space with norm $\|\cdot\|$ and let $\mathcal{L}(X, X)$ be the space of all bounded, linear operators mapping X into X.

   (a) For $T \in \mathcal{L}(X, X)$ give the definition of $\|T\|$.

   (b) Assume that $T \in \mathcal{L}(X, X)$ satisfies $\|I - T\| < 1$, where I is the identity operator. Prove that T is invertible and that $\sum_{n=0}^{\infty}(I - T)^n$ converges in $\mathcal{L}(X, X)$ to T^{-1}.

   (c) Assume that $T \in \mathcal{L}(X, X)$ is invertible and that $W \in \mathcal{L}(X, X)$ satisfies $\|S - T\| < \|T^{-1}\|^{-1}$. Prove that S is invertible.

8. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. Let $K(x, y)$ be a measurable function mapping $X \times Y$ into $\mathbb{R}$ with the following property. There exists a finite constant $M > 0$ such that for μ-almost every x

$$\int_Y |K(x, y)| \, d\nu(y) \leq M$$

and for ν-almost every y

$$\int_X |K(x, y)| \, d\mu(x) \leq M.$$

Prove that the operator

$$T : f \mapsto \int_{X \times Y} K(x, y) \, f(y) \, d\nu(y)$$

is a bounded operator from $L^p(Y)$ into $L^p(X)$ for all $1 \leq p \leq \infty$. Also prove that the operator norm of T does not exceed M.

Hint: For $1 < p < \infty$ first compute a suitable bound on $|Tf(x)|$ by applying Hölder's inequality to an appropriate factorization of the integrand.