DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

JANUARY 2018

Passing Standard: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

1. Show that there are no simples groups of order 80.

2. Determine the character table of the quaternion group

$$G = \{\pm 1, \pm i, \pm j, \pm k\}$$

of order 8, where the multiplication rule is determined by

$$i^2 = j^2 = k^2 = ijk = -1.$$

3. Let G be a finite group with center Z. Show that if G/Z is a p-group for some prime p, then G has a normal Sylow p-subgroup and p divides $\#Z$.

2. Commutative algebra

4. Let

$$R = \mathbf{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} \mid a, b \in \mathbf{Z}\} \subset \mathbf{C}.$$

Determine (with proof) whether R is a unique factorization domain.

5. Let V be a finite dimensional vector space over a field F and let $T : V \to V$ be an F-linear transformation. Let $f \in F[x]$ be the characteristic polynomial of T. Show that f is irreducible over F if and only if there are no proper nonzero subspaces $W \subseteq V$ such that $T(W) \subseteq W$.

6. Consider the ring

$$R = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \equiv b \pmod 3\}.$$

(1) Show that the homomorphism

$$f : \mathbf{Z}[x] \to R$$

sending 1 to $(1, 1)$ and x to $(3, 0)$ is surjective with kernel $(x^2 - 3x)$.

(2) Determine all prime ideals of R containing 5.

(3) Determine all prime ideals of R containing 3.

3. Field theory and Galois theory

7. Prove that -1 is not a sum of squares in the field $\mathbf{Q}(\alpha)$ where $\alpha \in \mathbf{C} \setminus \mathbf{R}$ is a cube root of 5.

8. Let $f(x) = x^4 - 4x^2 + 2$ and let K be the splitting field of $f(x)$ over $\mathbf{Q}$. Determine the Galois group of K over $\mathbf{Q}$.

9. Let K be a finite field and let L be an extension of K of degree n. Fix a monic irreducible polynomial $f \in K[x]$ of degree d dividing n. Show that there is $\alpha \in L$ which has minimal polynomial over K equal to f.