Math 331 **October 2009**

Differential Equations

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Midterm Exam

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1. Consider the equation

$$x\,y + \left(x^2 + \frac{1}{y}\right)\frac{dy}{dx} = 0.$$

   (a) Explain why the equation is not linear, separable or exact.

   (b) Multiply the equation by the integrating factor $\mu = 2y$, and find the general solution.

Solution:

(a) It is non-linear because of the $1/y$ term, and not separable because $(x^2 + 1/y)$ can't be factored. To check exactness: $M = x\,y$ and $N = x^2 + 1/y$, and we calculate

$$M_y = x \quad \textit{and} \quad N_x = 2\,x.$$

Since these differ, the equation isn't exact.

(b) Multiplying by $2\,y$, the equation becomes

$$2\,x\,y^2 + (2\,x^2\,y + 2)\frac{dy}{dx} = 0,$$

which is still not linear or separable. Again check exactness: set $M = 2\,x\,y^2$ and $N = 2\,x^2\,y + 2$, and compute

$$M_y = 4\,x\,y \quad \textit{and} \quad N_x = 4\,x\,y,$$

so the equation is now exact.
Thus there is some function $E(x,y)$ such that $E_x = M$ and $E_y = N$. Integrating, we get

$$E = \int M\,dx = \int 2\,x\,y^2\,dx = x^2\,y^2 + h(y).$$

Now differentiate in y: since $E_y = N$,

$$E_y = 2\,x^2\,y + h'(y) = N = 2\,x^2\,y + 2,$$

so $h'(y) = 2$ and $h(y) = 2\,y + C$. Thus the solution is

$$E(X,y) = x^2\,y^2 + 2\,y = K,$$

where the constant K is to be determined by the initial condition.

2. Solve the differential equation

$$t^2\, y'' = (y')^2$$

with initial values

$$y(1) = 1 \quad \text{and} \quad y'(1) = \frac{1}{2},$$

by first setting $v(t) = y'(t)$ and solving the appropriate equation for $v(t)$.

Solution:

Since $v(t) = y'(t)$, we get $v'(t) = y''(t)$, and so the equation becomes

$$t^2\, v' = v^2,$$

which is separable. Separating and integrating, we get

$$\int \frac{dv}{v^2} = \int \frac{dt}{t^2},$$

so that

$$\frac{-1}{v} = \frac{-1}{t} + K_1.$$

Now use the IC for v, namely $v(1) = y'(1) = \frac{1}{2}$, to get $K_1 = -1$. Thus we have

$$\frac{1}{v} = \frac{1}{t} + 1, \quad \textit{so} \quad v(t) = \frac{t}{1+t}.$$

Having found $v(t)$, we now find $y(t)$: since $v = y'$, we have

$$\begin{aligned} y(t) &= \int v(t)\, dt = \int \left(1 - \frac{1}{1+t}\right)\, dt \\ &= t - \log(1+t) + K, \end{aligned}$$

and we find K from the second IC $y(1) = 1$: we get $K = \log 2$ so the solution is

$$y(t) = t - \log(1+t) + \log 2.$$

3. When studying the formation of a vacuum in a simple gas, we obtain the equation

$$\frac{d}{dt}(c(t)\,t) + c(t) = 0$$

for the local sound speed $c(t)$. Find the general solution. [Hint: Set $y(t) = c(t)\,t$.]

Solution:

Setting $y(t) = c(t)\,t$ *we get* $c(t) = y(t)/t$, *so the equation becomes*

$$\frac{dy}{dt} + \frac{y}{t} = 0.$$

which is both separable and linear. Separating and integrating gives

$$\int \frac{dy}{y} = -\int \frac{dt}{t},$$

so that

$$\log y = -\log t + K_1 = \log(K\,t^{-1}), \quad \text{so} \quad y = \frac{K}{t}.$$

Now $c(t) = y(t)/t$, *so the general solution is*

$$c(t) = \frac{K}{t^2}.$$

4. A vat contains $120\,\ell$ of a solution containing $8\,kg$ of indigo. It is flushed out as follows: clean water is poured into the tank at a rate of $2\,\ell/min$; after one hour, the mixture is drained at a rate of $4\,\ell/min$, while clean water continues to pour in at $2\,\ell/min$. All flows are turned off when the fluid returns to its initial level. How much indigo is in the vat when the process is completed?

Solution:

For the first hour, only clean water enters and nothing leaves, so the volume increases but the amount of indigo is constant. Since the inflow is $2\,\ell/min$, after an hour the volume is $120 + 2 \cdot 60 = 240\,\ell$, and there is still $8\,kg$ of indigo.

We now model the draining process. Set $t = 0$, and let $V(t)$ denote the vloume of liquid and $I(t)$ the amount of indigo. We have

$$\frac{dV}{dt} = R_{in} - R_{out} = 2 - 4 = -2 \quad \textit{and} \quad V(0) = 240,$$

where R. denotes the volume flow rate, so we have

$$V(t) = 240 - 2\,t,$$

and V returns to 120 at $t = 60$ (one hour), as expected. Now $I(t)$ satisfies

$$\frac{dI}{dt} = c_{in}\,R_{in} - c_{out}\,R_{out} = 0 \cdot 2 - \frac{I}{V}\,4,$$

where c. is the concentraion (clean in, mixture out). So our equation is

$$\frac{dI}{dt} = -\frac{4\,I}{240 - 2\,t},$$

which is again linear and separable. Solving,

$$\int \frac{dI}{I} = 2 \int \frac{-dt}{120 - t},$$

so that

$$\log I(t) = 2\,\log(120 - t) + K = \log\left[C\,(120 - t)^2\right],$$

so that $I(t) = C\,(120 - t)^2$. Now $I(0) = 8$ so $C = 8/120^2$, and at time $t = 60$, we get

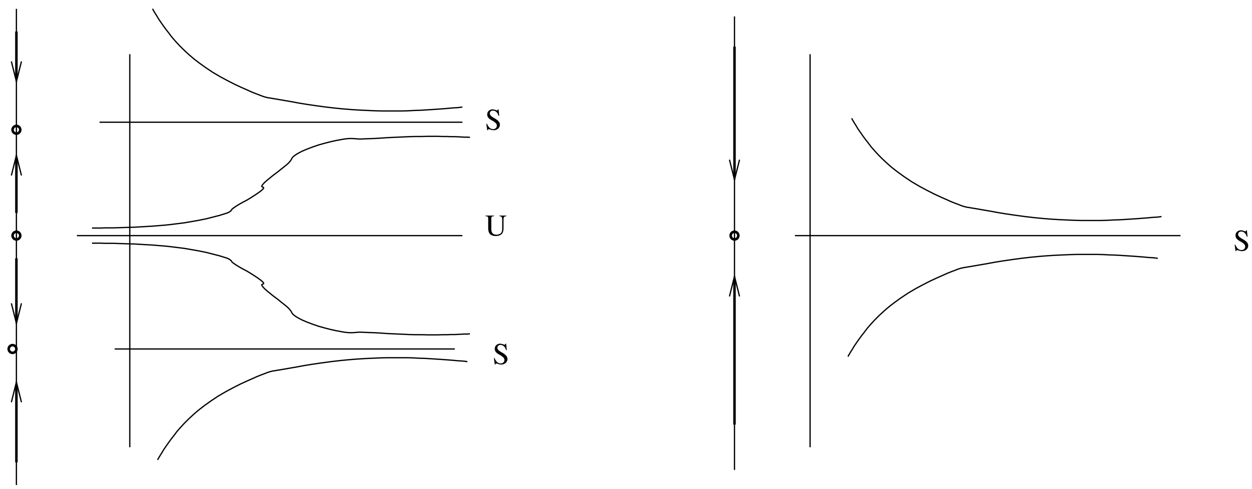
$$I(60) = \frac{8}{120^2}\,60^2 = \frac{8}{2^2} = 2\,kg.$$

5. Consider the equation

$$\frac{dy}{dt} = y\,(\beta - y^2),$$

which depends on a *parameter* β. Draw phase lines and a few trajectories for each of $\beta = -1/4$, 0 and 1/4. Describe the different features you see, especially the number (and type) of equilibria and the solutions near each equilibrium. Find the bifurcation point: that is, the value of β where the number of equilibria changes.

Solution:



The function $f_\beta(y) = y\,(\beta - y^2)$ is cubic, with roots $y = 0$ and $y = \pm\sqrt{\beta}$, which make sense only for $\beta > 0$. Phase lines and trajectories are shown: on the left is $\beta = \frac{1}{4}$, and the right is for $\beta = 0$ and $\beta = -\frac{1}{4}$ (only one equilibrium). Stable equilibria are labelled by an S; unstable by a U. Counting equilibria, there is one if $\beta \le 0$, three if $\beta > 0$; so the bifurcation occurs at $\beta = 0$.

Alternately, solve both $f_\beta(y) = 0$ and $f'_\beta(y) = 0$, that is

$$y\,(\beta - y^2) = 0 \quad \text{and} \quad \beta - 3\,y^2 = 0.$$

The second condition gives $\beta = 3\,y^2$, and plugging into the first gives $2\,y^3 = 0$, so $y = 0$ and $\beta = 0$.