KUMMER THEORY OF ABELIAN VARIETIES AND REDUCTIONS OF MORDELL-WEIL GROUPS

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Abstract. Let A be an abelian variety over a number field F with $\mathrm{End}_F A$ commutative. Let Σ be a subgroup of $A(F)$ and let x be a point of $A(F)$. Suppose that for almost all places v of F the reduction of x modulo v lies in the reduction of Σ modulo v. In this paper we prove that x must then lie in $\Sigma + A(F)_{\mathrm{tors}}$. This provides a partial answer to a generalization (by W. Gajda) of the support problem of Erdős.

Let A be an abelian variety over a number field F. We write $\mathrm{red}_v : A(F) \to A(k_v)$ for the reduction map at a place v of F with residue field k_v. W. Gajda has posed the following question.

Question. *Let Σ be a subgroup of $A(F)$. Suppose that x is a point of $A(F)$ such that $\mathrm{red}_v x$ lies in $\mathrm{red}_v \Sigma$ for almost all places v of F. Does it then follow that x lies in Σ?*

In this paper we use methods of Kummer theory to provide the following partial answer to this question.

Theorem. *Let A be an abelian variety over a number field F and assume that $\mathrm{End}_F A$ is commutative. Let Σ be a subgroup of $A(F)$ and suppose that $x \in A(F)$ is such that $\mathrm{red}_v x \in \mathrm{red}_v \Sigma$ for almost all places v of F. Then $x \in \Sigma + A(F)_{\mathrm{tors}}$.*

It does not appear that the torsion ambiguity can be eliminated with our present approach, and it is not clear to the author how to modify the arguments for the non-commutative case. We note that our theorem applies in particular to products of non-isogenous elliptic curves.

Gajda's question has its origins in the support problem of P. Erdős: if x and y are positive integers such that for any $n \geq 1$ the set of primes dividing $x^n - 1$ is the same as the set of primes dividing $y^n - 1$, then must x equal y? Corrales-Rodrigáñez and Schoof gave an affirmative answer to this question in [3] and also answered the corresponding question for elliptic curves; this was generalized by Banaszak, Gajda and Krasoń in

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[1] to certain abelian varieties with complex or real multiplication and $\mathrm{End}_F A$ a commutative maximal order. Recently Larsen [7] has given a proof of the support problem for arbitrary abelian varieties; see also [6] for results of Kowalaski on a closely related question. In this context the support problem takes the following form.

Question. *Let $x, y \in A(F)$ be non-torsion points. Suppose that the order of $\mathrm{red}_v x$ divides the order of $\mathrm{red}_v y$ for almost all places v of F. Does it follow that x and y satisfy an $\mathrm{End}_F A$-linear relation in $A(F)$?*

Taking $\Sigma = \mathrm{End}_F A\cdot y$, the support problem implies a weak form of our main theorem in the case that Σ is a cyclic $\mathrm{End}_F A$-module. The more precise question of Gajda we consider is one possible modification of the support problem for abelian varieties to a non-cyclic setting. The approach we use here is quite different from that of [3] and [1], relying more on the study of the Mordell-Weil group of A as a module for $\mathrm{End}_F A$ and less on Galois cohomology.

We give now an overview of our argument in the simplest case. Assume that A is simple, that $\mathcal{O} := \mathrm{End}_F A$ is integrally closed (so that it is a Dedekind domain), and that $A(F)$ is a free $\mathcal{O}$-module. With $\Sigma \subseteq A(F)$ and $x \in A(F)$ as in the theorem, it suffices to prove that $x \in \Sigma \otimes \mathbf{Z}_{(p)}$ for every prime p (with $\mathbf{Z}_{(p)}$ the localization of $\mathbf{Z}$ away from p). Fix, then, a prime p and suppose that $x \notin \Sigma \otimes \mathbf{Z}_{(p)}$. The first step, which is purely algebraic, is to show that under this assumption one can choose an $\mathcal{O}$-basis $y_1, \ldots, y_r$ of $A(F)$ such that $\psi_1(x) \notin \psi_1(\Sigma) + p^a\mathcal{O}$ for some $a > 0$; here $\psi_1 : A(F) \to \mathcal{O}$ is the projection onto the y_1-coordinate.

The next step is to choose an appropriate place v of F. We work instead over the extensions $F(A[p^n])$ of F. Using Kummer theory and the Chebatorev density theorem, we show that there is a $b > 0$ such that for any sufficiently large n there is a place w of $F(A[p^n])$ with $\mathrm{red}_w y_2, \ldots, \mathrm{red}_w y_r \in p^n A(k_w)$, while $\mathrm{red}_w y_1 \notin \mathfrak{p}_i^b A(k_w)$ for any i; here $p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_g^{e_g}$ is the ideal factorization of p in $\mathcal{O}$.

Fix $n \geq a + b$ and choose such a place w. By hypothesis we have $\mathrm{red}_w x = \mathrm{red}_w y$ for some $y \in \Sigma$. Expanding in terms of our chosen basis of $A(F)$, the choice of w implies that

$$\big(\psi_1(x) - \psi_1(y)\big)\,\mathrm{red}_w y_1 \in p^n A(k_w).$$

On the other hand, using the properties of ψ_1 and of w, one can show directly that

$$(\psi_1(x) - \psi_1(y))\,\mathrm{red}_w y_1 \notin p^{a+b} A(k_w).$$

As $n \geq a + b$, we have a contradiction, so that we must have had $x \in \Sigma \otimes \mathbf{Z}_{(p)}$. This completes our sketch of the argument in this case.

We now review the contents of this paper in more detail. We begin in Section 1.1 with a review of Kummer theory and in Section 1.2 we adapt the methods of Bashmakov–Ribet as in [9] to prove that the cokernel of the p-adic Kummer map is bounded. In Section 1.3 we discuss the relation between Kummer theory and reduction maps.

In the sketch above we assumed that $\mathcal{O}$ was an integrally closed domain and that $A(F)$ was free over $\mathcal{O}$. The algebra required to eliminate these assumptions is developed in Section 2. These results are combined with Kummer theory to produce places w as above in Section 3.1, and the proof of our main theorem is given in Section 3.2.

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1. Kummer theory

1.1. **Review of Kummer theory.** Let A be an abelian variety over a number field F; set $\mathcal{O} = \mathrm{End}_F A$. For $\alpha \in \mathcal{O}$ we set $F_\alpha = F(A[\alpha])$ and $G_\alpha = \mathrm{Gal}(F_\alpha/F)$. The *Kummer map*

$$\kappa_\alpha : A(F)/\alpha \to \mathrm{Hom}_{G_\alpha}\big(\mathrm{Gal}(\bar{F}/F_\alpha), A[\alpha]\big)$$

is defined as the composition

$$A(F)/\alpha \hookrightarrow H^1\big(F, A[\alpha]\big) \xrightarrow{\text{res}} H^1\big(F_\alpha, A[\alpha]\big)^{G_\alpha}$$

with the first map a coboundary map for the $\mathrm{Gal}(\bar{F}/F)$-cohomology of the Kummer sequence

$$0 \to A[\alpha] \to A(\bar{F}) \xrightarrow{\alpha} A(\bar{F}) \to 0$$

and the second map restriction to F_α. (Concretely, for $x \in A(F)$, $\kappa_\alpha(x)$ is the homomorphism sending $\gamma \in \mathrm{Gal}(\bar{F}/F_\alpha)$ to $\gamma(\frac{x}{\alpha}) - \frac{x}{\alpha} \in A[\alpha]$ where $\frac{x}{\alpha}$ is some fixed α^{th}-root of x in $A(\bar{F})$.)

If Γ is an $\mathcal{O}$-submodule of $A(F)$ and $\alpha \in \mathcal{O}$, we write $F_\alpha(\frac{1}{\alpha}\Gamma)$ for the extension of F_α generated by all α^{th}-roots of elements of Γ; alternately, $F_\alpha(\frac{1}{\alpha}\Gamma)$ is the fixed field of the intersection of the kernels of the homomorphisms $\kappa_\alpha(\Gamma)$. The Galois group $\mathfrak{g}_\alpha(\Gamma) := \mathrm{Gal}(F_\alpha(\frac{1}{\alpha}\Gamma)/F_\alpha)$ is an $\mathcal{O}[G_\alpha]$-module and κ_α restricts to an $\mathcal{O}$-linear map

$$\Gamma/\alpha \to \mathrm{Hom}_{G_\alpha}\big(\mathfrak{g}_\alpha(\Gamma), A[\alpha]\big).$$

We write the $\mathcal{O}[G_\alpha]$-dual of this map as

$$\lambda_\alpha^\Gamma : \mathfrak{g}_\alpha(\Gamma) \hookrightarrow \mathrm{Hom}_{\mathcal{O}}\big(\Gamma, A[\alpha]\big).$$

1.2. **p-adic Kummer theory.** Fix a rational prime p; set $\mathcal{O}_p = \mathcal{O}\otimes\mathbf{Z}_p$ and $K_p = \mathcal{O}\otimes\mathbf{Q}_p$. The Tate module $T_pA := \varprojlim A[p^n]$ (resp. Tate space $V_pA := T_pA \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$) is naturally an $\mathcal{O}_p[G_{p^\infty}]$-module (resp. $K_p[G_{p^\infty}]$-module) where $G_{p^\infty} = \mathrm{Gal}\big(F(A[p^\infty])/F\big)$. It follows from [8, Section 19, Corollary 2] that there is a decomposition

$$K_p = \prod M_{n_i} K_i \tag{1.1}$$

where $M_{n_i}K_i$ is the central simple algebra of $n_i \times n_i$-matrices over the division ring K_i. Corresponding to (1.1) is a decomposition $V_pA = \oplus V_iA^{n_i}$ of V_pA into $K_i[G_{p^\infty}]$-modules. By [5, Theorem 4], we have

$$\mathrm{End}_{\mathbf{Q}_p[G_{p^\infty}]} V_iA = K_i \tag{1.2}$$

for each i; in particular, each V_iA is an irreducible $K_i[G_{p^\infty}]$-module. We record a second immediate consequence of (1.2) in the next lemma.

Lemma 1.1. *Let Γ be an $\mathcal{O}$-module. Then the evaluation map*

$$\Gamma \otimes_{\mathcal{O}} K_i \to \mathrm{Hom}_{K_i[G_{p^\infty}]}\big(\mathrm{Hom}_{\mathcal{O}}(\Gamma, V_iA), V_iA\big)$$

is an isomorphism.

Fix an $\mathcal{O}$-submodule Γ of $A(F)$. The inverse limit $\mathfrak{g}_{p^\infty}(\Gamma)$ of the $\mathfrak{g}_{p^n}(\Gamma)$ is naturally an $\mathcal{O}_p[G_{p^\infty}]$-module endowed with an injection

$$\lambda^\Gamma_{p^\infty} : \mathfrak{g}_{p^\infty}(\Gamma) \hookrightarrow \mathrm{Hom}_{\mathcal{O}}(\Gamma, T_pA).$$

More generally, since $\mathcal{O}_p/p^n \cong \mathcal{O}/p^n$ for all n, for any $\mathcal{O}$-module $\Gamma \subseteq A(F) \otimes \mathbf{Z}_p$ we can still define $\mathfrak{g}_{p^n}(\Gamma)$ and $\lambda^\Gamma_{p^n}$ for $n \leq \infty$. In any case, there is a $K_p[G_{p^\infty}]$-module decomposition

$$\mathfrak{g}_{p^\infty}(\Gamma) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p = \oplus \mathfrak{g}_i(\Gamma)^{n_i} \tag{1.3}$$

(with n_i as in (1.1)) into $K_i[G_{p^\infty}]$-modules, and there are natural injections

$$\lambda^\Gamma_i : \mathfrak{g}_i(\Gamma) \hookrightarrow \mathrm{Hom}_{\mathcal{O}}(\Gamma, V_iA).$$

The decomposition (1.3) is functorial in the sense that there is a natural surjection $\mathfrak{g}_i(\Gamma) \twoheadrightarrow \mathfrak{g}_i(\Gamma')$ for any $\mathcal{O}$-submodule Γ' of Γ.

The main result of Kummer theory we need is the following. The proof is a straightforward adaptation of the methods of Bashmakov and Ribet.

Proposition 1.2. *Fix a rational prime p and let Γ be an $\mathcal{O}$-submodule of $A(F)$. Then the cokernel of $\lambda^\Gamma_{p^n}$ is bounded independent of n.*

Proof. First consider the cyclic case $\Gamma = \mathcal{O}\cdot x$ for $x \in A(F)$. If $\Gamma \cong \mathcal{O}$, then $\mathbf{Z}\cdot x$ is Zariski dense in A; the proposition thus follows from [2, Theorem 2] in this case. More generally, let A' denote the largest abelian subvariety of A, defined over F, in which $\mathbf{Z}\cdot x$ is Zariski dense;

set $\mathcal{O}' = \mathrm{End}_F A'$. Using the Poincaré reducibility theorem (see [8, Section 19, Theorem 1]), one checks easily that

$$\mathrm{Hom}_{\mathcal{O}}(\Gamma, V_p A) \cong \mathrm{Hom}_{\mathcal{O}'}(\Gamma, V_p A'),$$

so that the general cyclic case follows from [2, Theorem 2] applied to A'. In fact, one has $\mathrm{coker}\, \lambda_{p^\infty}^{\mathcal{O}x} = \mathrm{coker}\, \lambda_{p^\infty}^{\mathcal{O}x'}$ whenever $x, x' \in A(F)$ are sufficiently p-adically congruent, so that the same arguments apply for arbitrary $x \in A(F) \otimes \mathbf{Z}_p$.

For general Γ it suffices to show that each of the injections λ_i^Γ is an isomorphism. Suppose, then, that some λ_i^Γ is not surjective. Since $\mathrm{Hom}_{\mathcal{O}}(\Gamma, V_i A)$ is a direct sum of copies of the irreducible $K_i[G_{p^\infty}]$-module $V_i A$ (and thus in particular is a semisimple $K_i[G_{p^\infty}]$-module), it follows that there exists a $K_i[G_{p^\infty}]$-surjection

$$\varphi : \mathrm{Hom}_{\mathcal{O}}(\Gamma, V_i A) \twoheadrightarrow V_i A$$

annihilating $\mathfrak{g}_i(\Gamma)$. By Lemma 1.1 the map φ is given by evaluation at some $x \in \Gamma \otimes_{\mathcal{O}} K_i$; using the injection $K_i \hookrightarrow K_p$ and scaling φ if necessary, we may in fact assume that $x \in \Gamma \otimes \mathbf{Z}_p$. There is then a commutative diagram

$$\begin{array}{ccc} \mathfrak{g}_i(\Gamma) & \xhookrightarrow{\lambda_i^\Gamma} & \mathrm{Hom}_{\mathcal{O}}(\Gamma, V_i A) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \varphi \\ \mathfrak{g}_i(\mathcal{O}\cdot x) & \xhookrightarrow{\lambda_i^{\mathcal{O}x}} & V_i A \end{array}$$

The clockwise composition is zero by construction, so that we must have $\lambda_i^{\mathcal{O}x} = 0$ as well. By the cyclic case considered above this implies that x maps to zero in $\Gamma \otimes_{\mathcal{O}} K_i$. But then φ, which is evaluation at x, is also zero. This contradicts the surjectivity of φ and thus proves the proposition. $\square$

1.3. **Reductions and Frobenius elements.** We write k_w for the residue field of a finite extension F' of F at a place w and $\mathrm{red}_w : A(F') \to A(k_w)$ for the reduction map.

Lemma 1.3. *Fix $\alpha \in \mathcal{O}$ and $x \in A(F)$. Let w be a finite place of F_α, relatively prime to α, at which A has good reduction. Then $\mathrm{red}_w x$ lies in $\alpha A(k_w)$ if and only if $\lambda_\alpha^{\mathcal{O}\cdot x}(\mathrm{Frob}_w) = 0$, where $\mathrm{Frob}_w \in \mathrm{Gal}(F_\alpha(\frac{x}{\alpha})/F_\alpha)$ is the Frobenius element at w.*

Proof. Fix an α^{th}-root $\frac{x}{\alpha}$ of x in $A(\bar{F})$ and a place w' of $F_\alpha(\frac{x}{\alpha})$ over w. If $\lambda_\alpha^{\mathcal{O}\cdot x}(\mathrm{Frob}_w) = 0$, then w' is completely split over w so that $k_{w'} = k_w$. In particular, $\mathrm{red}_{w'} \frac{x}{\alpha} \in A(k_{w'})$ lies in $A(k_w)$; thus $\mathrm{red}_w x \in \alpha A(k_w)$ as claimed.

Conversely, if there is $y \in A(k_w)$ with $\alpha y = \mathrm{red}_w x$, then $y - \mathrm{red}_{w'} \frac{x}{\alpha}$ lies in $A[\alpha]$. Since y and $A[\alpha]$ are both in $A(k_w)$ we conclude that $\mathrm{red}_{w'} \frac{x}{\alpha}$ is in $A(k_w)$ as well. In particular, we have

$$\mathrm{Frob}_w\big(\mathrm{red}_{w'} \tfrac{x}{\alpha}\big) - \mathrm{red}_{w'} \tfrac{x}{\alpha} = 0. \tag{1.4}$$

On the other hand, $\mathrm{Frob}_w(\frac{x}{\alpha}) - \frac{x}{\alpha}$ already lies in $A[\alpha]$, which injects into $A(k_{w'})$; (1.4) thus forces

$$\mathrm{Frob}_w\big(\tfrac{x}{\alpha}\big) - \tfrac{x}{\alpha} = 0 \text{ in } A(\bar{F}).$$

This says exactly that $\lambda_\alpha^{\mathcal{O}\cdot x}(\mathrm{Frob}_w) = 0$, as claimed. □

We assume now that $\mathcal{O}$ is commutative. Suppose that $\mathfrak{a}$ is an ideal of $\mathcal{O}$ such that $\beta\mathfrak{a} \subseteq \alpha\mathcal{O}$ for some $\alpha, \beta \in \mathcal{O}$. Multiplication by β then yields a map $A[\alpha] \to A[\mathfrak{a}]$.

Lemma 1.4. *Let $\alpha, \beta, \mathfrak{a}$ be as above and fix $x \in A(F)$. Let w be a finite place of F_α, relatively prime to α, at which A has good reduction. If $\beta \cdot \lambda_\alpha^{\mathcal{O}\cdot x}(\mathrm{Frob}_w) \neq 0$, then $\mathrm{red}_w x \notin \mathfrak{a}A(k_w)$.*

Proof. We prove the contrapositive. Suppose that $\mathrm{red}_w x \in \mathfrak{a}A(k_w)$. Then

$$\beta\,\mathrm{red}_w x \in \beta\mathfrak{a}A(k_w) \subseteq \alpha A(k_w),$$

so that there is $y \in A(k_w)$ with $\beta\,\mathrm{red}_w x = \alpha y$. On the other hand, fixing an α^{th}-root $\frac{x}{\alpha}$ of x in $A(\bar{F})$ and a place w' of $F_\alpha(\frac{x}{\alpha})$ lying above w, we also have $\beta\,\mathrm{red}_w x = \alpha\beta\,\mathrm{red}_{w'} \frac{x}{\alpha}$. Therefore

$$y - \beta\,\mathrm{red}_{w'} \tfrac{x}{\alpha} \in A[\alpha].$$

From here the argument proceeds as in the second half of the proof of Lemma 1.3 above to show that $\beta \cdot \lambda_\alpha^{\mathcal{O}\cdot x}(\mathrm{Frob}_w) = 0$. □

We remark that the converse of Lemma 1.4 holds in the case that $\alpha\mathcal{O} = \mathfrak{a}\mathfrak{a}'$ with $\mathfrak{a}, \mathfrak{a}'$ relatively prime and $\beta \in \mathfrak{a}' \cap (1 - \mathfrak{a})$.

2. Modules over commutative, reduced, finite, flat **Z**-algebras

2.1. **Projections.** Let $\mathcal{O}$ be a commutative, reduced, finite, flat $\mathbf{Z}$-algebra. The normalization $\tilde{\mathcal{O}}$ of $\mathcal{O}$ decomposes as a product $\prod_{j=1}^h \tilde{\mathcal{O}}_j$ of Dedekind domains. (See [4, Section 11.2], for example, for a discussion of the normalization of a reduced ring.) We say that a $\mathbf{Z}$-linear map $t : \mathcal{O} \to \mathbf{Z}$ is *full* if it is non-trivial on $\mathcal{O} \cap \tilde{\mathcal{O}}_j$ for each j. Note that such a map always exists; indeed, this is clear for $\tilde{\mathcal{O}}$ (simply take the sum of the trace maps $\tilde{\mathcal{O}}_j \to \mathbf{Z}$), and multiplying a full map for $\tilde{\mathcal{O}}$ by $[\tilde{\mathcal{O}} : \mathcal{O}]$ yields a full map $\mathcal{O} \to \mathbf{Z}$.

Lemma 2.1. *Fix a full map $t : \mathcal{O} \to \mathbf{Z}$. Then the map*

$$\text{(2.1)} \qquad \begin{aligned} \mathrm{Hom}_{\mathcal{O}}(N, \mathcal{O}) &\to \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z}) \\ f &\mapsto t \circ f \end{aligned}$$

has finite cokernel for any finitely generated $\mathcal{O}$-module N.

Proof. Since $\mathcal{O}$ has finite index in $\tilde{\mathcal{O}}$, it suffices to prove the result after replacing $\mathcal{O}$ by $\tilde{\mathcal{O}}$ and N by $N \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$. We may therefore assume that $\mathcal{O}$ decomposes as a product $\prod \mathcal{O}_i$ of Dedekind domains. There is then a corresponding decomposition $N = \oplus N_i$, and by the definition of a full map it suffices to prove the lemma for each factor N_i; that is, we may assume that $\mathcal{O}$ is a Dedekind domain.

In this case every finitely generated $\mathcal{O}$-module has a free submodule of finite index; this allows one to reduce to the case that N is free, and then to the case that N is free of rank one. (2.1) is then a map

$$\text{(2.2)} \qquad \mathcal{O} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, \mathcal{O}) \to \mathrm{Hom}_{\mathbf{Z}}(\mathcal{O}, \mathbf{Z})$$

between two free $\mathbf{Z}$-modules of the same rank, so that it suffices to prove that it is injective. For this, note that (2.2) is $\mathcal{O}$-linear; thus its kernel is an ideal of $\mathcal{O}$. However, every non-zero ideal of $\mathcal{O}$ has finite index and $\mathrm{Hom}_{\mathbf{Z}}(\mathcal{O}, \mathbf{Z})$ is torsion-free; therefore (2.2) must be either zero or injective. As t itself lies in the image, it is obviously non-zero. ☐

We now fix a finitely generated $\mathcal{O}$-module N and a $\mathbf{Z}$-submodule M of N containing the $\mathbf{Z}$-torsion submodule N_{tors} of N.

Lemma 2.2. *Fix $x \in N$ and suppose that p is a rational prime such that $x \notin M \otimes \mathbf{Z}_{(p)}$. Then there is an $\mathcal{O}$-linear map $\psi : N \to \mathcal{O}$ such that $\psi(x) \notin \psi(M) + p^n\mathcal{O}$ for sufficiently large n.*

Proof. Choose a $\mathbf{Z}$-basis $y_1, \ldots, y_r \in N$ of N/N_{tors} such that there are integers $d_1, \ldots, d_r$ with

$$M = \langle d_1 y_1, \ldots, d_r y_r \rangle \oplus N_{\text{tors}}.$$

(Of course, some of the d_i may be zero.) Writing $x = a_1 y_1 + \cdots + a_r y_r + t$ with $a_i \in \mathbf{Z}$ and $t \in N_{\text{tors}}$, the fact that $x \notin M \otimes \mathbf{Z}_{(p)}$ implies that there is some index i such that

$$\text{(2.3)} \qquad \mathrm{ord}_p\, a_i < \mathrm{ord}_p\, d_i.$$

Let $\psi_0 : N \to \mathbf{Z}$ be $\#N_{\text{tors}}$ times projection onto y_i; this is a well-defined map, and it follows from (2.3) that $\psi_0(x) \notin \psi_0(M) + p^n\mathbf{Z}$ for sufficiently large n. (In fact, $n > \mathrm{ord}_p(a_i \cdot \#N_{\text{tors}})$ suffices.)

Fix a full map $t : \mathcal{O} \to \mathbf{Z}$. By Lemma 2.1, we can find a non-zero integer b such that $b\psi_0$ is in the image of (2.1). Thus there is an $\mathcal{O}$-linear map $\psi : N \to \mathcal{O}$ with $b\psi_0 = t \circ \psi$. Since $t(p^n\mathcal{O}) \subseteq p^n\mathbf{Z}$, we conclude that $\psi(x) \notin \psi(M) + p^n\mathcal{O}$ for sufficiently large n, as desired. □

2.2. **Pre-bases.** We continue with $M \subseteq N$ as before. Fix $y \in N$ not in N_{tors} and let $\varphi : \mathcal{O} \twoheadrightarrow \mathcal{O}\cdot y$ be the $\mathcal{O}$-linear surjection sending 1 to y. We define $\eta_0(y)$ to be the least positive integer m such that there exists an $\mathcal{O}$-linear map $\psi : \mathcal{O}\cdot y \to \mathcal{O}$ with the composition

$$\mathcal{O}\cdot y \xrightarrow{\psi} \mathcal{O} \xrightarrow{\varphi} \mathcal{O}\cdot y$$

multiplication by m. (Let K_j denote the fraction field of $\tilde{\mathcal{O}}_j$; since $\mathcal{O} \otimes \mathbf{Q} = \prod K_j$, to see that any maps ψ as above exist it suffices to prove the corresponding fact after replacing $\mathcal{O}$ by $\prod K_j$. In this context the map φ identifies with the quotient map

$$\prod K_j \to \prod_{j\in J} K_j$$

for some non-empty subset J of $\{1, \ldots, h\}$, so that the existence of ψ is obvious.)

We say that $y_1, \ldots, y_r \in N$ are an *$\mathcal{O}$-pre-basis* of N if:

- $y_i \notin N_{\mathrm{tors}}$ for all i;
- $(\mathcal{O}\cdot y_1) \oplus \cdots \oplus (\mathcal{O}\cdot y_r)$ injects into N with finite cokernel.

(Note that we do not require that the corresponding map $\mathcal{O}^r \to N$ is injective.) Let $\eta'(y_1, \ldots, y_r)$ be the order of this cokernel and define

$$\eta(y_1, \ldots, y_r) = \eta'(y_1, \ldots, y_r) \cdot \eta_0(y_1) \cdots \eta_0(y_r).$$

It then follows from the definition of $\eta_0(y_i)$ that there are $\mathcal{O}$-linear maps

$$\psi_i^{y_1,\ldots,y_r} : N \to \mathcal{O}$$

for $i = 1, \ldots, r$ such that

$$\eta(y_1, \ldots, y_r)y = \psi_1^{y_1,\ldots,y_r}(y)y_1 + \cdots + \psi_r^{y_1,\ldots,y_r}(y)y_r \tag{2.4}$$

for all $y \in N$. We usually just write η and ψ_i if the pre-basis $y_1, \ldots, y_r$ is clear from context. A standard inductive procedure shows that pre-bases always exist.

Proposition 2.3. *Fix $x \in N$ and suppose that p is a rational prime such that $x \notin M \otimes \mathbf{Z}_{(p)}$. Then there is an $\mathcal{O}$-pre-basis $y_1, \ldots, y_r$ of N such that $\psi_1(x) \notin \psi_1(M) + p^n\mathcal{O}$ for sufficiently large n.*

Proof. By Lemma 2.2, we may choose an $\mathcal{O}$-linear map $\psi : N \to \mathcal{O}$ such that $\psi(x) \notin \psi(M) + p^n\mathcal{O}$ for sufficiently large n. Let K' denote the image of $\psi \otimes \mathbf{Q}$; we have $K' = \prod_{j\in J} K_j$ for some non-empty subset J of $\{1, \ldots, h\}$. In particular, K' is a projective $\prod K_j$-module, so that there exists a map $\varphi_0 : K' \to N \otimes \mathbf{Q}$ such that $(\psi \otimes \mathbf{Q}) \circ \varphi_0$ is the identity on K'. Scaling φ_0 by an integer we obtain an $\mathcal{O}$-linear map

$\varphi : \tilde{\mathcal{O}}' \to N$ such that $\psi \circ \varphi$ is multiplication by some non-zero integer; here $\tilde{\mathcal{O}}' = \prod_{j \in J} \tilde{\mathcal{O}}_j$.

Set $y_1 = \varphi(1)$ and choose an $\mathcal{O}$-pre-basis $y_2, \ldots, y_r$ for $\ker \psi$. Then $y_1, \ldots, y_r$ is an $\mathcal{O}$-pre-basis of N and $\psi_1 = m\psi$ for some non-zero integer m. It thus follows from the definition of ψ that $\psi_1(x) \notin \psi_1(M) + p^n\mathcal{O}$ for sufficiently large n, as desired. □

2.3. **Ideals.** We continue with $\mathcal{O}$ as above. Fix a rational prime p and write the $\mathbf{Z}$-exponent of $\tilde{\mathcal{O}}/\mathcal{O}$ as cp^d with $d \geq 0$ and c relatively prime to p. Let

$$p\tilde{\mathcal{O}} = \tilde{\mathfrak{p}}_1^{e_1} \cdots \tilde{\mathfrak{p}}_g^{e_g}$$

be the factorization of $p\tilde{\mathcal{O}}$ into prime ideals of $\tilde{\mathcal{O}}$; for each $i \in \{1, \ldots, g\}$ we let $\mu_p(i)$ denote the unique $j \in \{1, \ldots, h\}$ such that $\tilde{\mathfrak{p}}_i$ is the pullback of a prime ideal on $\tilde{\mathcal{O}}_j$. For $y \in N$ we define $I_p(y) \subseteq \{1, \ldots, g\}$ to be the set of indices i such that the image of y in $N \otimes_{\mathcal{O}} \tilde{\mathcal{O}}_{\tilde{\mathfrak{p}}_i}$ is non-torsion. In fact, since every proper ideal of each $\tilde{\mathcal{O}}_j$ has finite index, we have

$$I_p(y) = \{i; \operatorname{rank}_{\mathbf{Z}}\big((\mathcal{O} \cap \tilde{\mathcal{O}}_{\mu_p(i)})\cdot y\big) > 0\}. \tag{2.5}$$

For $i = 1, \ldots, g$ and any n, we define ideals of $\mathcal{O}$ by

$$\mathfrak{p}_{i,n} = \tilde{\mathfrak{p}}_i^{e_i n} \cap \mathcal{O}.$$

The reader is invited to focus on the case $d = 0$, when $\mathfrak{p}_{i,n} = \mathfrak{p}_{i,1}^n$ and the analysis below is quite a bit simpler. In the general case, we have $cp^d\tilde{\mathfrak{p}}_i^{e_i n} \subseteq \mathfrak{p}_{i,n}$; since the $\tilde{\mathfrak{p}}_i$ are relatively prime, it follows that

$$c^{g-1}p^{d(g-1)}\mathcal{O} \subseteq \mathfrak{p}_{i,n} + \prod_{j \neq i} \mathfrak{p}_{j,n} \tag{2.6}$$

for all n. Furthermore, $p^n\tilde{\mathcal{O}} \cap \mathcal{O} \subseteq p^{n-d}\mathcal{O}$ for $n \geq d$, so that

$$p^n\mathcal{O} \subseteq \mathfrak{p}_{1,n} \cap \cdots \cap \mathfrak{p}_{g,n} \subseteq p^{n-d}\mathcal{O}; \tag{2.7}$$

$$c^g p^{n+dg}\mathcal{O} \subseteq \mathfrak{p}_{1,n} \cdots \mathfrak{p}_{g,n} \subseteq p^{n-d}\mathcal{O}; \tag{2.8}$$

for any $n \geq d$.

Lemma 2.4. *Let N be a finitely generated $\mathcal{O}$-module. Fix $\alpha \in \mathcal{O}$ and $x \in N$. Suppose that there is an index i and non-negative integers a, b such that:*

(1) *$\alpha \notin \mathfrak{p}_{i,a}$;*
(2) *$x \notin \mathfrak{p}_{i,b}N$;*
(3) *$N[p^{a+d}] \subseteq p^b N$.*

Then $\alpha x \notin p^{a+b+d}N$.

Proof. We first replace $\mathcal{O}$ by $\varprojlim \mathcal{O}/\mathfrak{p}_{i,n}$, N by $\varprojlim N/\mathfrak{p}_{i,n}$, and $\tilde{\mathcal{O}}$ by $\varprojlim \tilde{\mathcal{O}}/\tilde{\mathfrak{p}}_i^n$. Let $\tilde{\mathfrak{p}}$ denote the maximal ideal of $\tilde{\mathcal{O}}$, so that $\tilde{\mathfrak{p}}^{e_i} = p\tilde{\mathcal{O}}$; set $\mathfrak{p}_n = \tilde{\mathfrak{p}}^{e_i n} \cap \mathcal{O}$. With this notation we have $\alpha \notin \mathfrak{p}_a$ and $x \notin \mathfrak{p}_b N$, and it suffices to prove that $\alpha x \notin p^{a+b+d}N$. Note that $\alpha \notin \tilde{\mathfrak{p}}^{e_i a}$, so that there is some $\beta \in \tilde{\mathcal{O}}$ with $\alpha\beta = p^a$.

Set $C = \tilde{\mathcal{O}}/\mathcal{O}$ and $\tilde{N} = N \otimes_{\mathcal{O}} \tilde{\mathcal{O}}$; C is killed by p^d and there is an exact sequence

$$\operatorname{Tor}_1^{\mathcal{O}}(N, C) \to N \xrightarrow{\iota} \tilde{N} \to N \otimes_{\mathcal{O}} C \to 0. \tag{2.9}$$

Suppose now that $\alpha x \in p^{a+b+d}N$. Applying ι and multiplying by β, we find that $p^a\iota(x) \in p^{a+b+d}\tilde{N}$. By (2.9) we have $p^d\tilde{N} \subseteq \iota(N)$, so that this implies that $p^a x - p^{a+b}n \in \ker\iota$ for some $n \in N$. Again by (2.9) this kernel is killed by p^d; we conclude that

$$p^{a+d}x \in p^{a+b+d}N.$$

Thus

$$x \in p^bN + N[p^{a+d}] \subseteq p^bN \subseteq \mathfrak{p}_bN.$$

Since $x \notin \mathfrak{p}_bN$ by hypothesis, this yields the desired contradiction. □

3. Reductions of Mordell-Weil groups

3.1. **Galois elements.** Let A be an abelian variety over a number field F. By [8, Section 19, Corollary 2] the ring $\mathcal{O} := \operatorname{End}_F$ is a reduced, finite, flat $\mathbf{Z}$-algebra. We further assume that it is commutative; we fix a rational prime p, and we continue with the notations of Section 2 for this ring $\mathcal{O}$ and prime p. By (2.6) we may fix $a_{i,n} \in \mathfrak{p}_{i,n}$ and $b_{i,n} \in \prod_{j\neq i}\mathfrak{p}_{j,n}$ such that $a_{i,n} + b_{i,n} = c^{g-1}p^{d(g-1)}$. The map

$$\begin{aligned}\varphi_n : A[p^{n-d}] &\to A[\mathfrak{p}_{1,n}] \oplus \cdots \oplus A[\mathfrak{p}_{g,n}]\\ t &\mapsto (b_{1,n}t, \ldots, b_{g,n}t)\end{aligned}$$

is then well-defined by (2.8).

Lemma 3.1. *The cokernel of φ_n is bounded independent of n.*

Proof. Since $p^n \in \mathfrak{p}_{i,n}$ we can define a map

$$\begin{aligned}\psi_n : A[\mathfrak{p}_{1,n}] \oplus \cdots \oplus A[\mathfrak{p}_{g,n}] &\to A[p^{n-d}]\\ (t_1, \ldots, t_g) &\mapsto p^d(t_1 + \cdots + t_g).\end{aligned}$$

As $c^{g-1}p^{d(g-1)} - b_{i,n} \in \mathfrak{p}_{i,n}$, the map $\varphi_n \circ \psi_n$ is just multiplication by $c^{g-1}p^{dg}$. The lemma follows from this. □

For an $\mathcal{O}$-submodule Γ of $A(F)$, we now write

$$\lambda^{\Gamma}_{\mathfrak{p}_{i,n+d}} : \mathfrak{g}_{p^n}(\Gamma) \to \operatorname{Hom}_{\mathcal{O}}\big(\Gamma, A[\mathfrak{p}_{i,n+d}]\big)$$

for the composition of $\lambda^{\Gamma}_{p^n}$ with φ_{n+d} and projection to $A[\mathfrak{p}_{i,n+d}]$. In the next lemma we use the natural map $\mathfrak{g}_{p^n}(\Gamma) \to \mathfrak{g}_{p^m}(\Gamma)$ (corresponding to multiplication by p^{n-m} from $\operatorname{Hom}_{\mathcal{O}}(\Gamma, A[\mathfrak{p}_{i,n+d}])$ to $\operatorname{Hom}_{\mathcal{O}}(\Gamma, A[\mathfrak{p}_{i,m+d}])$) to regard $\lambda^{\Gamma}_{\mathfrak{p}_{i,m+d}}$ as a map from $\mathfrak{g}_{p^n}(\Gamma)$ for $n \geq m$.

Lemma 3.2. *Let $y_1, \dots, y_r$ be an $\mathcal{O}$-pre-basis of $A(F)$. Then there is an integer b such that for all sufficiently large n there is a $\sigma_n \in \mathfrak{g}_{p^n}(A(F))$ with*

$$\lambda^{\mathcal{O}\cdot y_j}_{p^n}(\sigma_n) = 0 \textit{ for } j = 2, \dots, r;$$

$$\lambda^{\mathcal{O}\cdot y_1}_{\mathfrak{p}_{i,b}}(\sigma_n) \neq 0 \textit{ for all } i \in I_p(y_1).$$

Proof. The cokernel of the natural map

$$\pi : \operatorname{Hom}_{\mathcal{O}}\big(A(F), A[p^n]\big) \to \bigoplus_{j=1}^{r} \operatorname{Hom}_{\mathcal{O}}\big(\mathcal{O}\cdot y_j, A[p^n]\big)$$

is bounded independent of n by the definition of a pre-basis. Combined with Proposition 1.2, it follows that the cokernel of

$$\pi \circ \lambda^{A(F)}_{p^n} : \mathfrak{g}_{p^n}\big(A(F)\big) \to \bigoplus_{j=1}^{r} \operatorname{Hom}_{\mathcal{O}}\big(\mathcal{O}\cdot y_j, A[p^n]\big)$$

is bounded independent of n. Finally, by Lemma 3.1 we conclude that the cokernel of the map

$$\mathfrak{g}_{p^n}\big(A(F)\big) \to \Big(\bigoplus_{i \in I_p(y_1)} \operatorname{Hom}_{\mathcal{O}}\big(\mathcal{O}\cdot y_1, A[\mathfrak{p}_{i,n+d}]\big)\Big) \oplus \Big(\bigoplus_{j=2}^{r} \operatorname{Hom}_{\mathcal{O}}\big(\mathcal{O}\cdot y_j, A[p^n]\big)\Big) \tag{3.1}$$

is bounded independent of n.

By the definition of the set $I_p(y_i)$, for each $i \in I_p(y_1)$ there is some $m > 0$ such that $p^{n+d-m} \operatorname{Hom}_{\mathcal{O}}(\mathcal{O}{\cdot}y_1, A[\mathfrak{p}_{i,n+d}]) \neq 0$ for sufficiently large n. (That is, these groups grow with n.) Since the cokernel of (3.1) is bounded, it follows that there is an integer b such that for sufficiently large n there is $\sigma_n \in \mathfrak{g}_{p^n}(A(F))$ with

$$\sigma_n|_{\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}y_j, A[p^n])} = 0 \text{ for } j = 2, \dots, r;$$

$$p^{n+d-b}\sigma_n|_{\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}y_1, A[\mathfrak{p}_{i,n+d}])} \neq 0 \text{ for all } i \in I_p(y_1).$$

By the remarks preceding the lemma, this σ_n is the required element of $\mathfrak{g}_{p^n}(A(F))$. ∎

Lemma 3.3. *Let $y_1, \dots, y_r$ be an $\mathcal{O}$-pre-basis of $A(F)$. Then there is an integer b such that for all sufficiently large n there are infinitely many places w of F_{p^n} with*

$$\mathrm{red}_w y_j \in p^n A(k_w) \text{ for } j = 2, \dots, r;$$

$$\mathrm{red}_w y_1 \notin \mathfrak{p}_{i,b} A(k_w) \text{ for } i \in I_p(y_1).$$

Proof. Let n be sufficiently large and fix σ_n as in Lemma 3.2. If w is a place of F_{p^n} with $\mathrm{Frob}_w = \sigma_n$ in $\mathfrak{g}_{p^n}(A(F))$, then w satisfies the conditions of the lemma by Lemmas 1.3 and 1.4. Since the Chebatorev density theorem guarantees the existence of infinitely many such w, the lemma follows. □

3.2. **Reduction of subgroups.** We are now in a position to prove our main result.

Proposition 3.4. *Let A be an abelian variety over a number field F; assume that $\mathcal{O} = \mathrm{End}_F A$ is commutative. Fix a rational prime p and let Σ be a subgroup of $A(F)$ containing $A(F)_{\mathrm{tors}}$. Suppose that $x \in A(F)$ is such that*

$$\mathrm{red}_v x \in \mathrm{red}_v \Sigma \tag{3.2}$$

for almost all places v of F. Then x lies in $\Sigma \otimes \mathbf{Z}_{(p)}$.

Proof. Suppose that $x \notin \Sigma \otimes \mathbf{Z}_{(p)}$. By Proposition 2.3 we can then choose an $\mathcal{O}$-pre-basis $y_1, \dots, y_r$ of $A(F)$ such that there is an integer a with

$$\psi_1(x) \notin \psi_1(\Sigma) + p^a \mathcal{O}. \tag{3.3}$$

Let b be the integer determined by $y_1, \dots, y_r$ in Lemma 3.3 and fix $n > a + b + 2d$. Let w be a place of F_{p^n} as in Lemma 3.3; by (3.2) we may further assume that there is a $y \in \Sigma$ with $\mathrm{red}_w x = \mathrm{red}_w y$. Multiplying by η, by (2.4) we have

$$\psi_1(x)\,\mathrm{red}_w y_1 + \cdots + \psi_r(x)\,\mathrm{red}_w y_r = \psi_1(y)\,\mathrm{red}_w y_1 + \cdots + \psi_r(y)\,\mathrm{red}_w y_r.$$

Thus

$$\big(\psi_1(x) - \psi_1(y)\big)\,\mathrm{red}_w y_1 \in p^n A(k_w) \tag{3.4}$$

by the definition of w.

Set $\alpha = \psi_1(x) - \psi_1(y)$; by (3.3) and (2.7), $\alpha \notin \mathfrak{p}_{i,a+d}$ for some i. Fix such an i. Since $\alpha \in \mathrm{im}\,\psi_1$, by (2.5) we have $i \in I_p(y_1)$; thus we also have $\mathrm{red}_w y_1 \notin \mathfrak{p}_{i,b} A(k_w)$ by the definition of w. Since $A(k_w)[p^{a+2d}] \subseteq p^b A(k_w)$ (as $A[p^n] \subseteq A(k_w)$ and $a+b+2d < n$), we may therefore apply Lemma 2.4 to conclude that $\alpha\,\mathrm{red}_w\, y_1 \notin p^{a+b+2d} A(k_w)$. Since $a + b + 2d < n$, this contradicts (3.4), and thus proves the proposition. □

Corollary 3.5. *Let A be an abelian variety over a number field F and assume that* $\mathrm{End}_F A$ *is commutative. Let Σ be a subgroup of $A(F)$ containing $A(F)_{\mathrm{tors}}$ and suppose that $x \in A(F)$ is such that* $\mathrm{red}_v x \in \mathrm{red}_v \Sigma$ *for almost all places v of Σ. Then $x \in \Sigma$.*

Proof. This is immediate from Proposition 3.4 applied for all primes p. □

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