

MATH 791N ALGEBRAIC NUMBER THEORY HW 2 + ε

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- (1) #12, #14, #15, #17, #18, #24, #26, #27 from Farshid's HW2.
- (2) Let  $K$  be a number field with ring of integers  $\mathcal{O}$ . Let  $\mathfrak{p}$  be a prime ideal of  $\mathcal{O}$  containing the rational prime  $p$ . Show that if  $\mathfrak{p}$  divides a rational integer  $a$ , then  $p$  divides  $a$  in  $\mathbf{Z}$ .
- (3) Let  $u$  be a unit in  $\mathbf{Z}[\zeta_p]$ . Recall that  $u/\bar{u} = \pm\zeta_p^a$  for some  $a$ . Show that in fact the sign is  $+$ . (Hint: Expand  $u$  in terms of  $\zeta_p$  and compare  $u$  and  $\bar{u}$  modulo  $1 - \zeta_p$ .)
- (4) Recall that for an extension  $L/K$  of number fields and a prime ideal  $\mathfrak{p}_L$  of  $L$  sitting over a prime ideal  $\mathfrak{p}_K$  of  $K$ , we define

$e(\mathfrak{p}_L/\mathfrak{p}_K) =$  exponent of  $\mathfrak{p}_L$  appearing in the factorization of  $\mathfrak{p}_K \cdot \mathcal{O}_L$

$$f(\mathfrak{p}_L/\mathfrak{p}_K) = [\mathcal{O}_L/\mathfrak{p}_L : \mathcal{O}_K/\mathfrak{p}_K].$$

Now Let  $M/L/K$  be number fields. Fix a prime  $\mathfrak{p}_K$  of  $K$  and let  $\mathfrak{p}_L$  (resp.  $\mathfrak{p}_M$ ) be a prime of  $L$  (resp.  $M$ ) above  $\mathfrak{p}_K$  (resp.  $\mathfrak{p}_L$ ). Prove that

$$e(\mathfrak{p}_M/\mathfrak{p}_K) = e(\mathfrak{p}_M/\mathfrak{p}_L) \cdot e(\mathfrak{p}_L/\mathfrak{p}_K).$$

$$f(\mathfrak{p}_M/\mathfrak{p}_K) = f(\mathfrak{p}_M/\mathfrak{p}_L) \cdot f(\mathfrak{p}_L/\mathfrak{p}_K).$$

- (5) Let  $L$  and  $M$  be extensions of the number field  $K$  such that  $M/K$  is unramified everywhere. Prove that  $ML/L$  is unramified everywhere.
- (6) Define the *Bernoulli numbers*  $B_n$  by

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

- (a) Prove that  $B_n = 0$  for  $n > 1$  odd. (Hint: write  $\frac{t}{e^t - 1}$  as the sum of an even function and a linear function.)
- (b) Prove that

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0.$$

Conclude that  $B_k \in \mathbf{Q}$ .

(7) (a) Prove that

$$t \cot t = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{(2n)!} t^{2n}.$$

(Hint: rewrite the even function of (6a) in a way in which you can use the expressions for  $\sin t$  and  $\cos t$  in terms of complex exponentials.)

(b) Use the identity

$$\frac{\sin t}{t} = \prod_{k=1}^{\infty} \left( 1 - \frac{t^2}{(k\pi)^2} \right)$$

to prove that

$$\cot t = \frac{1}{t} + 2t \sum_{k=1}^{\infty} \frac{1}{t^2 - k^2\pi^2}.$$

(c) Recall that the Riemann  $\zeta$ -function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

Prove that

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n} B_{2n}}{2(2n)!} \pi^{2n}$$

for  $n \geq 1$ .