Problem Set 10

(1) In this problem you will give a proof of the fundamental theorem of algebra. Throughout the problem we identify S^1 with the unit circle in the complex plane $\mathbf{C}$. Let

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$$

be a polynomial with complex coefficients; set

$$q(z) = a_{n-1}z^{n-1} + \cdots + a_0.$$

We assume that $p(z)$ has no roots in $\mathbf{C}$ and derive a contradiction.

(a) For any real r define a map

$$f_r(z) : S^1 \to S^1$$

$$f_r(z) = \frac{p(rz)}{|p(rz)|}$$

(Note that f_r is well-defined since $p(rz) \neq 0$ for all r, z.) Prove that f_r is homotopic to $f_{r'}$ for any $r, r' \geq 0$.

(b) Let $\gamma_n : S^1 \to S^1$ denote the map given by $\gamma_n(z) = z^n$. Prove that for r sufficiently large, f_r is homotopic to γ_n. (Hint: Choose r large enough so that $|rz|^n > |q(rz)|^n$ for all $z \in S^1$ and then slowly kill off the $q(z)$ term in $p(z)$.) What does this tell you about the homomorphism

$$f_{r*} : \pi_1(S^1, 1) \to \pi_1(S^1, f_r(1))$$

for r sufficiently large?

(c) Prove that the homomorphism

$$f_{0*} : \pi_1(S^1, 1) \to \pi_1(S^1, f_0(1))$$

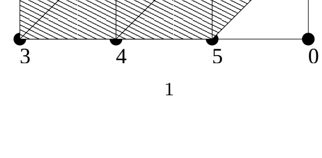
is the zero map. Explain why these three facts give a contradiction.

(d) Now let $p(z)$ be as above, but no longer assume that $p(z)$ has no zeros. The function f_r is still well-defined for all but finitely many values of r. Explain what happens to the homomorphisms f_{r*} as r increases. Please don't try too hard to prove anything.

(2) Fix relatively prime integers p, q. Define an action of $\mathbf{Z}/p$ on $S^3 = \{(z_0, z_1) \in \mathbf{C}^2\,;\, |z_0|^2 + |z_1|^2 = 1\}$ by $g(z_0, z_1) = (e^{2\pi i/p}z_0, e^{2\pi q i/p}z_1)$ where g is a generator of $\mathbf{Z}/p$. Define the Lens space $L(p, q)$ as the quotient of S^3 by this $\mathbf{Z}/p$-action. Show that this action is properly discontinuous; conclude that $L(p, q)$ has fundamental group $\mathbf{Z}/p$.

(3) Let G be a finitely generated abelian group. Prove that there exists a path connected topological manifold X with $\pi_1(X, x) \cong G$. (Hint: use the previous problem and the structure theorem for finitely generated abelian groups.)

(4) (a) Compute the fundamental group of the following simplicial complex, which is a triangulation of the mobius strip with a disc removed.



(b) Use your calculations above and the Seifert–Van Kampen theorem to compute the fundamental group of the space X obtained by starting with a mobius strip and replacing a disc by a second mobius strip (glued in along its boundary circle).

(5) Let X be a path connected topological space and let

$$f : [0,1] \times [0,1] \to X$$

be a continuous map. Note that for any path

$$\gamma : [0,1] \to [0,1] \times [0,1]$$

we obtain by composition of maps a path $f \circ \gamma : [0,1] \to X$.

(a) Let $\gamma_1, \gamma_2 : [0,1] \to [0,1] \times [0,1]$ be paths. Prove that $f \circ \gamma_1$ and $f \circ \gamma_2$ are homotopic.

(b) Suppose also that $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$. Prove that γ_1 and γ_2 are homotopic relative to $\{0,1\} \subseteq [0,1]$.

(6) Let G be a path connected topological group with identity e.

(a) Let $i : G \to G$ denote the inversion map. Prove that the homomorphism $i_* : \pi_1(G,e) \to \pi_1(G,e)$ is given by $\gamma \mapsto \gamma^{-1}$. (Hint: you need to show that $\gamma \cdot i_*\gamma$ is homotopic to the constant map at e. Find a way to do this by applying the previous problem to the map $f : [0,1] \times [0,1] \to G$ given by $f(s,t) = \gamma(s) \cdot \gamma(t)^{-1}$.)

(b) Use the above fact to give a proof that $\pi_1(G,e)$ is abelian. (Hint: when is the inversion map a homomorphism of a group?)

(7) We say that a topological space X is a *curve* if it is a compact connected topological manifold of dimension 1 (without boundary). (Recall that the last condition means that every point $x \in X$ has an open neighborhood $U \subseteq X$ which is homeomorphic to the open interval $(0,1)$.) Note that under this definition neither of $(0,1)$ or $[0,1]$ is a curve.

Formulate and prove a classification theorem for curves; that is, find a list of curves such that any curve is homeomorphic to a unique member of your list. You may want to follow the following strategy.

(a) Show that any curve X has a finite open cover $X = U_1 \cup \cdots \cup U_n$ where each U_i is equipped with a homeomorphism $\varphi_i : U_i \to (0,1)$, and such that $\varphi_i(U_i \cup U_j)$ is either empty or else of the form $(0,\varepsilon)$ or $(\varepsilon,1)$ for some $\varepsilon \in (0,1)$.

(b) Show that X is homeomorphic to the geometric realization of the simplicial complex with vertices (i,j) for any i,j such that $U_i \cap U_j$ is nonempty, and such that (i,j) and (i',j') are connected by an edge if and only if $\{i,j\} \cap \{i',j'\}$ is nonempty.

(c) Show that any simplicial complex with geometric realization a topological 1-manifold must have geometric realization homeomorphic to the circle.