MATH 300.1 WORKSHEET 3

Chapters 4 and 6:

Problems involving Mathematical Induction.

1. Prove that $1 + 2 + 3 + \cdots + n = \frac{1}{2}(n + 1)n$.
2. Prove that for any $x \neq \pm 1$, $1 + x^2 + x^4 + \cdots + x^{2n} = \frac{x^{2n+2}-1}{(x+1)(x-1)}$.
3. If p is a prime number, use Math Induction on n and the Binomial Theorem to show that $n^p \equiv n \pmod{p}$ for all $n = 1, 2, \cdots$.
4. A sequence of integers $x_1, x_2, x_3, \cdots$, is defined by $x_1 = 3, x_2 = 7$ and the recursion

$$x_k = 5x_{k-1} - 6x_{k-2}, \forall k \geq 3.$$

Show that $x_n = 2^n + 3^{n-1}$ for all $n \geq 1$.

Functions, Cardinality and Permutations.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by the formula $f(x) = x^2 - 1$.
(a) What is the domain of f?
(b) What is the codomain of f?
(c) What is the image of f?
(d) Find maximal subsets X, Y of $\mathbb{R}$ such that $f : X \to Y$ is a bijection. Find a formula for the inverse $f^{-1} : Y \to X$.

6. Let $f : \mathbb{Z}_7 \to \mathbb{Z}_7$ be the function $f([x]) = [x^2]$ and let $g : \mathbb{Z}_7 \to \mathbb{Z}_7$ be the function $g([x]) = [x^3]$. Find a formula for $g \circ f$. Sketch the graphs of f, g and $g \circ f$. Which one of them are bijections?
7. For which values of $n \in \mathbb{Z}_7$ is $f_n : \mathbb{Z}_7 \to \mathbb{Z}_7$, defined by $f_n([x]) = [x^4 + nx]$, a bijection? Find the inverse f_n^{-1} for each bijection.
8. Show that $\#(\mathbb{P}_2 \times \mathbb{P}_n) = \#\mathbb{P}_{2n}$ for any $n = 1, 2, \cdots$ by constructing an explicit bijection $f : \mathbb{P}_2 \times \mathbb{P}_n \to \mathbb{P}_{2n}$.
9. Let X be the set consisting of integers which have remainder 1 when divided by 3. Show that $\#X = \#\mathbb{Z}$.
10. A permutation $\sigma \in S_5$ is given as a composition of transpositions

$$\sigma = (12) \circ (23) \circ (34) \circ (45).$$

Find out what is σ.