

PUTNAM 2009 WEEK 4: NUMBER THEORY

Basic number theory concepts to remember: modular arithmetic, unique factorization, greatest common divisor (a,b) can be written as a linear combination $ax+by$, Chinese Remainder Theorem, positional notation, Fermat's little theorem $a^p \equiv a \pmod{p}$.

Easier Problems

1. Suppose the number $n = 47^{99}$ is written in base ten. What is the last (rightmost) digit of n ? How about the last two digits of n ?
2. If $2n + 1$ and $3n + 1$ are both perfect squares, show that $40|n$.
3. Let n be a positive integer. Suppose that 2^n and 5^n begin with the same digit. Then there is only one possible value for this common initial digit. Find, with proof, that digit.
4. Let $1, 2, 3, \dots, 2007, 2008, 2009, 2010, 2012, 2015, \dots$ be a sequence defined by $x_k = k$ for $k = 1, 2, \dots, 2009$ and $x_{k+1} = x_k + x_{k-2008}$ for $k \geq 2009$. Show that the sequence has 2008 consecutive terms each divisible by 2009.
5. Let $f(n)$ denote the sum of the digits of n (when n is expressed in base 10). For any integer n , prove that eventually the sequence

$$f(n), f(f(n)), f(f(f(n))), \dots$$

will become constant. Call this constant value the *digital root* of n . For example, the digital root of 392 is 5.

Harder Problems

6. Is a natural number uniquely determined by the product of its (positive) divisors?

7. Show that there exist three consecutive integers, each of which is divisible by a 2009th power of an integer (not including 1^{2009} and $(-1)^{2009}$).

8. Prove that the digital root (see Problem 5) of the product of any two twin primes (i.e., primes that differ by 2), other than 3 and 5, is 8.

9. Consider two lists. List A consists of the positive powers of 10 (10, 100, 1000, ...) written in base 2. List B consists of the positive powers of 10 written in base 5. Show that, for any integer $n > 1$, there is exactly one number in exactly one of the lists that is exactly n digits long.

Powers of 10	List A	List B
10	1010 (4 digits)	20 (2 digits)
100	1100100 (7 digits)	400 (3 digits)
1000	1111101000 (10 digits)	13000 (5 digits)
10000	10011100010000 (14 digits)	310000 (6 digits)

10. Let p be a prime number. Let $h(x)$ be a polynomial with integer coefficients such that $h(0), h(1), \dots, h(p^2 - 1)$ are distinct modulo p^2 . Show that $h(0), h(1), \dots, h(p^3 - 1)$ are distinct modulo p^3 .