PUTNAM 2009 WEEK 4: REAL ANALYSIS

Here are some useful results from Real Analysis. Of course you don't have to remember them all to solve problems that follow. Still, these theorems have an astonishing number of applications.

Riemann Sums: if a function is Riemann-integrable, e.g. if it is continuous on a closed finite interval, then the integral is the limit of the Riemann sums.

The Intermediate Value Theorem: If f is continuous on $[a, b]$, then every value between $f(a)$ and $f(b)$ is of the form $f(c)$ for $a \leq c \leq b$.

The Extreme Value Theorem: A continuous function on a compact set (for example, on the interval $[0, 1]$) attains its sup and inf.

Rolle's Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then $f'(u) = 0$ for some point $u \in (a, b)$.

The Mean Value Theorem: If f is continuous on $[a, b]$ and differentiable on (a, b), and $f(a) = f(b)$, then there is a point $u \in (a, b)$ at which $f'(u) = \frac{f(b)-f(a)}{b-a}$.

Convergence: A bounded monotone sequence converges. A sum in which the entries have alternating sign and which decrease in absolute value converges. A monotone sum whose corresponding integral is bounded converges (the integral comparison test). A sequence bounded above and below by two other convergent sequences must converge (the squeeze principle).

Taylor's Formula with Remainder: If h has continuous n-th derivatives, then for any $x > 0$ there exists $\theta_n \in [0, x]$ such that

$$h(x) = h(0) + h'(0)x + \cdots + h^{(n-1)}(0)x^{n-1}/(n-1)! + h^{(n)}(\theta_n)x^n/n!.$$

Easier Problems

1. Show that right now, there are two diametrically-opposed points on the earth's equator that have exactly the same temperature.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that

$$f(x, y) + f(y, z) + f(z, x) = 0$$

for all real numbers x, y, and z. Prove that there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x, y) = g(x) - g(y)$$

for all real numbers x and y.

3. (a) Suppose $f(x)$ is a polynomial of odd degree with real coefficients. Then $f(x) = 0$ has a real root. (b) Any real square matrix with an odd number of rows has a real eigenvalue. (c) Euler's Theorem: Any orthogonal linear transformation of $\mathbb{R}^3$ that preserves orientation is a rotation around some axis.

4. Prove that

$$\binom{2k}{k} = \frac{2}{\pi}\int_0^{\pi/2} (2\sin x)^{2k}\,dx$$

5. Bernoulli's inequality: $(1+x)^a \geq 1 + ax$ for $x > -1$ and $a \geq 1$.

6. What happens if you put a random positive number in your (infinite precision) calculator, and repeatedly hit the sequence of buttons "$1/x$", "+" and "1"? (In other words, what happens if you iterate $x \mapsto 1/x + 1$?).

7. Compute

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2}.$$

Harder Problems

8. Suppose f is differentiable on $(-\infty, \infty)$ and there is a constant $k < 1$ such that $|f'(x)| \leq k$ for all real x. Show that the equation $f(x) = x$ has a unique solution.

9. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 1, 2, 3, \ldots$.

10. Is $\sqrt{2}$ the limit of a sequence of numbers of the form

$$\sqrt[3]{n} - \sqrt[3]{m}, \quad n, m = 0, 1, 2, \ldots?$$

11. What is the largest possible radius of a circle contained in a 4-dimensional hypercube of side length 1?

12. Let f be continuous and monotonically increasing, with $f(0) = 0$ and $f(1) = 1$. Prove that

$$f(1/10)+f(2/10)+\cdots+f(9/10)+f^{-1}(1/10)+f^{-1}(2/10)+\cdots+f^{-1}(9/10) \leq 99/10.$$