PUTNAM 2009 WEEK 3: COMBINATORICS

Easier Problems

1. Recall that binomial coefficients $\binom{n}{k}$ form the Pascal triangle

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

where each number is equal to the sum of two numbers *above* it (why?). Now consider a triangle

$$\begin{array}{ccccccccc} &&&&\frac{1}{1}&&&& \\ &&&\frac{1}{2}&&\frac{1}{2}&&& \\ &&\frac{1}{3}&&\frac{1}{6}&&\frac{1}{3}&& \\ &\frac{1}{4}&&\frac{1}{12}&&\frac{1}{12}&&\frac{1}{4}& \\ \frac{1}{5}&&\frac{1}{20}&&\frac{1}{30}&&\frac{1}{20}&&\frac{1}{5} \end{array}$$

where each number is equal to the sum of two numbers *below* it. Find out how numbers in this triangle are related to binomial coefficients.

2.[1] Show that $\binom{n+m}{k} = \binom{n}{k}\binom{m}{0} + \binom{n}{k-1}\binom{m}{1} + \binom{n}{k-2}\binom{m}{2} + \ldots + \binom{n}{0}\binom{m}{k}$.

3. Show that $\binom{n}{0} + \binom{n+1}{1} + \cdots + \binom{n+k}{k} = \binom{n+k+1}{k}$.

4. Find the number of ways to write a positive integer n as a sum of positive integers. Sums with differently ordered terms are considered different. For example, if $n = 3$ then there are exactly 4 possible sums:

$$3,\ 1+1+1,\ 1+2,\ 2+1$$

5. Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

[1]Many problems in this set can be solved using the following basic idea: Two sets have the same number of elements if there is a bijection between them. For example, $\binom{n}{k} = \binom{n}{n-k}$ because the number of ways to choose k objects out of n is equal to the number of ways to choose a complementary subset of $n-k$ elements. Of course finding a bijection can be tricky.

Harder Problems

6. Any infinite sequence of real numbers contains either an infinite increasing subsequence or an infinite decreasing subsequence.

7. Find all $n \geq 0$ such that $\binom{n}{k}$ is odd for any $0 \leq k \leq n$.

8. Let S be a set of points in the plane and let R be a fixed positive real number. We want to place discs of radius R on the plane such that,

(i) Every point in S is covered by exactly one disc, and
(ii) Ever disc is centered at a point in S.

This may or may not be possible, but if it is possible, show that all solutions use the exact same number of discs.

9. For positive integers m and n, let $f(m,n)$ denote the number of n-tuples $(x_1, x_2, \ldots, x_n)$ of integers such that $|x_1| + |x_2| + \cdots + |x_n| \leq m$. Show that $f(m,n) = f(n,m)$.

10. Suppose we have a necklace of n beads. Each bead is labelled with an integer and the sum of all these labels is $n-1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k-1 \qquad \text{for} \quad k = 1, 2, \ldots, n.$$

11. Let σ be a permutation of numbers $\{1, \ldots, n\}$. We call σ bad if it contains a decreasing subsequence $\sigma_{i_1} > \ldots > \sigma_{i_{10}}$ for some $i_1 < \ldots < i_{10}$. Prove that the number of good permutations is at most 81^n. Use this to show that the probability that a random permutation is good tends to 0 as $n \to \infty$.