PUTNAM 2009 WEEK 1: INDUCTION AND PIGEONHOLE

Don't solve problems that you already know how to solve. Work in groups. Try small cases. Do examples. Look for patterns. Use lots of paper. Talk it over. Choose effective notation. Try the problem with different numbers. Work backwards. Argue by contradiction. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. Ask.

Easier Problems

1. Given a picture of 100 circles, prove that it is possible to color regions in black and white so that any two regions that have a common border are colored differently.
2. Consider the sequence defined by $a_1 = 1$ and $a_n = \sqrt{2 + a_{n-1}}$. Prove that $a_n < 2$ for all n.
3. Find the maximal number of regions into which 50 lines can divide the plane.
4. Fibonacci numbers are defined by

$$F_0 = 1, \quad F_1 = 1, \quad \ldots, \quad F_{n+2} = F_{n+1} + F_n.$$

Prove that any two consecutive Fibonacci numbers are coprime.
5. In how many ways can a $2 \times n$ rectangle be tiled with 2×1 dominoes? (Hint: experiment with small n first).
6. Prove that in a room with n people, at least two people know exactly the same number of people. Assume knowing is a symmetric relation: If Paul knows Pete, then Pete knows Paul.
7. On a certain street there are twenty houses, ten along each side of the road. Abe, Bill, Cathy, Dierdre, and Ed each live in one of the houses. Prove that among these five people, there must be two of them who live on the same side of the street separated by no more than three houses between them.

Harder Problems

10. You have n coins $C_1, C_2, \ldots, C_n$. For each k, C_k is biased so that, when tossed, it has probability $1/(2k+1)$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n.
10. Prove that given any five points on a sphere, there exists a closed hemisphere which contains four of the points.
9. Choose 51 positive integers from 1 to 100. Prove that one of them is a multiple of another.)
8. For each non-empty subset of $\{1, 2, ..., n\}$ take the sum of the elements divided by the product. Show that the sum of the resulting quantities is $n^2 + 2n - (n+1)s_n$, where $s_n = 1 + 1/2 + 1/3 + ... + 1/n$.
11. The first $2n$ natural numbers are arbitrarily divided into two groups of n numbers each. The numbers in the first group are sorted in ascending order, i.e., $a_1 < ... < a_n$, and the numbers in the second group are sorted in descending order: $b_1 > ... > b_n$. Find, with proof, the sum $|a_1 - b_1| + ... + |a_n - b_n|$.
12. A graph is called simple if any pair of vertices is connected by at most one edge and no edge connects a vertex to itself. Prove that a simple graph with v vertices and no "triangles" (triples of pairwise connected vertices) can have at most $v^2/4$ edges.