ALGEBRA 612. FINAL EXAM.

Name: ____

The test contains three sections (Galois Theory, Commutative Algebra, and Representations of Finite Groups). Each section contains three problems. You have to choose **two problems from each section**. Only these problems will be graded. A problem with multiple parts counts as one problem. Please select your six problems here:

Staple your solutions to this problem set. Textbooks or lecture notes are not allowed. Problems are ordered randomly. Good luck!

1. Galois Theory

In this section k denotes an arbitrary field.

1. Let G and H be finite groups of automorphisms of the field k. Show that $k^G = k^H$ if and only if $G = H$.

2. Let $K = k(u, v)$ be a field generated by algebraically independent elements u and v. Let $F = k(u^2, v^2) \subset K$. Show that there exists infinitely many intermediate subfields $F \subset L \subset K$ if and only if $\operatorname{char} k = 2$.

3. Let $f(x) \in \mathbb{Q}[x]$ be a polynomial of degree 4 with roots $\alpha_1, \ldots, \alpha_4$. Let

$$u = (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4),$$
$$v = (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4),$$
$$w = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_3).$$

(a) Show that u, v, w are roots of a cubic polynomial with rational coefficients (do not derive explicit formulas for these coefficients). (b) Explain how to compute $\alpha_1, \ldots, \alpha_4$ in radicals explicitly using u, v, w.

2. Commutative Algebra

All rings in this section are commutative with 1.

4. Let $p_1, \ldots, p_r$ be primes (not necessarily different) and let $n_1, \ldots, n_r$ be positive integers. Consider the ring

$$R = (\mathbb{Z}/p_1^{n_1}\mathbb{Z}) \times \ldots \times (\mathbb{Z}/p_r^{n_r}\mathbb{Z}).$$

Compute $\operatorname{Spec} R$ and compute the localization $R_{\mathfrak{p}}$ for each prime ideal $\mathfrak{p}$.

5. Let k be an algebraically closed field and let $R = k[x_1, \ldots, x_n]$. Let $I \subset R$ be a radical ideal and let $J \subset R$ be any ideal. Show that

$$\overline{V(I) - V(J)} = V((I : J)),$$

where

$$(I : J) = \{r \in R \,|\, rJ \subset I\}$$

and $\overline{X}$ denotes the Zariski closure of X.

6. Let $R \subset S$ be integral domains such that R is a UFD and S is integral over R. Let $K \subset F$ be their fields of fractions. Let $s \in S \subset F$ and let $f(x) \in K[x]$ be its monic minimal polynomial. Show that all coefficients of $f(x)$ belong to R.

3. Representations of Finite Groups

All representations in this section are complex representations.

7. Let $\mathbb{F}_q$ be an arbitrary finite field and let G be the group of affine transformations of $\mathbb{F}_q$ of the form $x \mapsto ax + b$, where $a \in \mathbb{F}_q^*$ and $b \in \mathbb{F}_q$. (a) List dimensions of all complex irreducible representations of G (including how many irreducible representations are there in each dimension). (b) Since G acts on $\mathbb{F}_q$, it also acts on functions $\mathbb{F}_q \to \mathbb{C}$. Let ρ be a $(q-1)$-dimensional complex representation of G on functions $f : \mathbb{F}_q \to \mathbb{C}$ such that $\sum\limits_{x \in \mathbb{F}_q} f(x) = 0$. Show that ρ is irreducible and compute its character.

8. Let p be a prime number and let G be a group of matrices of the form

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}, \qquad x, y, z \in \mathbb{F}_p.$$

(a) List dimensions of all complex irreducible representations of G (including how many irreducible representations are there in each dimension). (b) Let

$$z_0 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G.$$

Show that z_0 acts in any p-dimensional representation of G as a scalar operator $\zeta\mathrm{Id}$, where ζ is one of the primitive p-th roots of 1. (c) Compute characters of p-dimensional irreducible representations of G (it is not necessary to know their explicit models for this problem).

9. Let G_1 and G_2 be two finite groups and let V_1 and V_2 be their complex representations. (a) Show that $V_1 \otimes V_2$ is a representation of $G_1 \times G_2$, where this group acts by formula

$$(g_1, g_2)(v_1 \otimes v_2) = (g_1 v_1) \otimes (g_2 v_2).$$

What can you say about the character of this representation? (b) Show that if V_1 and V_2 are irreducible representations of G_1 and G_2 then $V_1 \otimes V_2$ is an irreducible representation of $G_1 \times G_2$. (c) Prove that any irreducible representation of $G_1 \times G_2$ appears that way.