ALGEBRA: LECTURE NOTES

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CONTENTS

§1. CATEGORIES AND FUNCTORS

§1.1. **Categories.** Most mathematical theories deal with situations when there are some maps between objects. The set of objects is usually somewhat static (and so boring), and considering maps makes the theory more dynamic (and so more fun). Usually there are some natural restrictions on what kind of maps should be considered: for example, it is rarely interesting to consider any map from one group to another: usually we require this map to be a homomorphism.

The notion of a category was introduced by Samuel Eilenberg and Saunders MacLane to capture situations when we have both objects and maps between objects (called morphisms). This notion is slightly abstract, but extremely useful. Before we give a rigorous definition, here are some examples of categories:

EXAMPLE 1.1.1.

- The category **Sets**: objects are sets, morphisms are arbitrary functions between sets.
- **Groups**: objects are groups, morphisms are homomorphisms.
- **Ab**: objects are Abelian groups, morphisms are homomorphisms.

- **Rings**: objects are rings, morphisms are homomorphisms of rings. Often (for example in this course) we only consider commutative rings with identity.
- **Top**: topological spaces, morphisms are continuous functions.
- **Mflds**: objects are smooth manifolds, morphisms are differentiable maps between manifolds.
- $\mathbf{Vect}_k$: objects are k-vector spaces, morphisms are linear maps.

Notice that in all these examples we can take compositions of morphisms and (even though we rarely think about this) composition of morphisms is associative (because in all these examples morphisms are functions with some restrictions, and composition of functions between sets is certainly associative). The associativity of composition is a sacred cow of mathematics, and essentially the only axiom required to define a category:

DEFINITION 1.1.2. A category C consists of the following data:

- The set of objects $\mathbf{Ob}(C)$. Instead of writing "X is an object in C", we can write $X \in \mathbf{Ob}(C)$, or even $X \in C$.
- The set of morphisms $\mathbf{Mor}(C)$. Each morphism f is a morphism *from* an object $X \in C$ *to* an object $Y \in C$. More formally, $\mathbf{Mor}(C)$ is a disjoint union of subsets $\mathbf{Mor}(X, Y)$ over all $X, Y \in C$. It is common to denote a morphism by an arrow $X \xrightarrow{f} Y$.
- There is a composition law for morphisms
$$\mathbf{Mor}(X, Y) \times \mathbf{Mor}(Y, Z) \to \mathbf{Mor}(X, Z), \qquad (f, g) \mapsto g \circ f$$
which takes $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ to the morphism $X \xrightarrow{g \circ f} Z$.
- For each object $X \in C$, we have an *identity morphism* $X \xrightarrow{\mathrm{Id}_X} X$.

These data should satisfy the following basic axioms:

- The composition law is associative.
- The composition of any morphism $X \xrightarrow{f} Y$ with $X \xrightarrow{\mathrm{Id}_X} X$ (resp. with $Y \xrightarrow{\mathrm{Id}_Y} Y$) is equal to f.

Here is another example.

EXAMPLE 1.1.3. Let G be a group. Then we can define a category C with just one object (let's denote it by O) and with
$$\mathbf{Mor}(C) = \mathbf{Mor}(O, O) = G.$$
The composition law is just the composition law in the group and the identity element Id_O is just the identity element of G.

DEFINITION 1.1.4. A morphism $X \xrightarrow{f} Y$ is called an *isomorphism* if there exists a morphism $Y \xrightarrow{g} X$ (called an inverse of f) such that
$$f \circ g = \mathrm{Id}_Y \qquad \text{and} \qquad g \circ f = \mathrm{Id}_X\,.$$

In the example above, every morphism is an isomorphism. Namely, an inverse of any element of $\mathbf{Mor}(C) = G$ is its inverse in G.

A category where any morphism is an isomorphism is called a *groupoid*, because any groupoid with one object can be obtained from a group G as above. Indeed, axioms of the group (associativity, existence of a unit, existence of an inverse) easily translate into axioms of the groupoid (associativity of the composition, existence of an identity morphism, existence of an inverse morphism).

Of course not any category with one object is a groupoid and not any groupoid has one object.

EXAMPLE 1.1.5. Fix a field k and a positive integer n. We can define a category C with just one object (let's denote it by O) and with

$$\mathbf{Mor}(C) = \mathrm{Mat}_{n,n}\,.$$

The composition law is given by the multiplication of matrices. The identity element Id_O is just the identity matrix. In this category, a morphism is an isomorphism if and only if the corresponding matrix is invertible.

Here is an example of a category with a different flavor:

EXAMPLE 1.1.6. Recall that a *partially ordered set*, or a *poset*, is a set I with an order relation $\preceq$ which is

- reflexive: $i \preceq i$ for any $i \in I$,
- transitive: $i \preceq j$ and $j \preceq k$ implies $i \preceq k$, and
- anti-symmetric: $i \preceq j$ and $j \preceq i$ implies $i = j$.

For example, we can take the usual order relation $\leq$ on real numbers, or divisibility relation $a|b$ on natural numbers ($a|b$ if a divides b). Note that in this last example not any pair of elements can be compared.

Interestingly, we can view any poset as a category C. Namely, $\mathbf{Ob}(C) = I$ and for any $i, j \in I$, $\mathbf{Mor}(i,j)$ is an empty set if $i \not\preceq j$ and $\mathbf{Mor}(i,j)$ is a set with one element if $i \preceq j$. The composition of morphisms is defined using transitivity of $\preceq$: if $\mathbf{Mor}(i,j)$ and $\mathbf{Mor}(j,k)$ is non-empty then $i \preceq j$ and $j \preceq k$, in which case $i \preceq k$ by transitivity, and therefore $\mathbf{Mor}(i,k)$ is non-empty. In this case $\mathbf{Mor}(i,j)$, $\mathbf{Mor}(j,k)$, and $\mathbf{Mor}(i,k)$ consist of one element each, and the composition law $\mathbf{Mor}(i,j) \times \mathbf{Mor}(j,k) \to \mathbf{Mor}(i,k)$ is defined in a unique way.

Notice also that, by reflexivity, $i \preceq i$ for any i, hence $\mathbf{Mor}(i,i)$ contains a unique morphism: this will be our identity morphism Id_i.

Here is an interesting example of a poset: let X be a topological space. Let I be the set of open subsets of X. This is a poset, where the order relation is the inclusion of open subsets $U \subset V$. The corresponding category can be denoted by $\mathbf{Top}(X)$.

§1.2. **Functors.** If we want to consider several categories at once, we need a way to relate them! This is done using functors.

DEFINITION 1.2.1. A covariant (resp. contravariant) functor F from a category C to a category D is a rule that, for each object $X \in C$, associates an object $F(X) \in D$, and for each morphism $X \xrightarrow{f} Y$, associates a morphism $F(X) \xrightarrow{F(f)} F(Y)$ (resp. $F(Y) \xrightarrow{F(f)} F(X)$). Two axioms have to be satisfied:

- $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$ for any $X \in C$.

- F preserves composition: for any $X \xrightarrow{g} Y$ and $Y \xrightarrow{f} Z$, we have $F(f \circ g) = F(f) \circ F(g)$ (if F is covariant) and $F(f \circ g) = F(g) \circ F(f)$ (if F is contravariant).

EXAMPLE 1.2.2. Let's give some examples of functors.

- Inclusion of a subcategory, for example we have a functor
$$\mathbf{Ab} \to \mathbf{Groups}$$
that sends any Abelian group G to G (considered simply as a group) and that sends any homomorphism $G \xrightarrow{f} H$ of Abelian groups to f (considered as a homomorphism of groups).
- More generally, we have all sorts of *forgetful* covariant functors $C \to D$. This simply means that objects (and morphisms) of C are objects (and morphisms) of D with some extra data and some restrictions on this data. The forgetful functor simply 'forgets' about this extra data. For example, there is a forgetful functor $\mathbf{Vect}_k \to \mathbf{Sets}$ that sends any vector space to the set of its vectors and that sends any linear map to itself (as a function from vectors to vectors). Here we 'forget' that we can add vectors, multiply them by scalars, and that linear maps are linear!
- Here is an interesting contravariant functor: the duality functor $\mathbf{Vect}_k \to \mathbf{Vect}_k$ sends any vector space V to the vector space V^* of linear functions on V. A linear map $L : V \to U$ is sent to a contragredient linear map $L^* : U^* \to V^*$ (which sends a linear function $f \in U^*$ to a linear function $v \mapsto f(L(v))$ in V^*).
- A very important contravariant functor is a functor $\mathbf{Top} \to \mathbf{Rings}$ that sends any topological space X to its ring of continuous functions $C^0(X, \mathbb{R})$ and that sends any continuous map $X \xrightarrow{f} Y$ to a *pull-back homomoprhism* $f^* : C^0(Y, \mathbb{R}) \to C^0(X, \mathbb{R})$ (just compose a function on Y with f to get a function on X).
- Here is an interesting variation: let's fix a topological space X and consider a functor $\mathbf{Top}(X) \to \mathbf{Rings}$ that sends any open subset $U \subset X$ to continuous functions $C^0(U, \mathbb{R})$ on U. For any inclusion $U \subset V$ of open sets, the pull-back homomorphism $C^0(V, \mathbb{R}) \to C^0(U, \mathbb{R})$ is just restriction: we restrict a function on V to a function on U. This functor $\mathbf{Top}(X) \to \mathbf{Rings}$ is an example of a *sheaf*.

§1.3. **Equivalence of Categories.** It is tempting to consider a category of all categories with functors as morphisms! Indeed, we can certainly define a composition of two functors $C \xrightarrow{F} D$ and $D \xrightarrow{G} E$ in an obvious way, and we have obvious identity functors $C \xrightarrow{\mathrm{Id}_C} C$ that do not change either objects or morphisms. There are some slight set-theoretic issues with this super-duper category, but we are going to ignore them.

However, one interesting issue here is when should we consider two categories C and D as equivalent? An obvious approach is to say that C and D are isomorphic categories if there exist functors $C \xrightarrow{F} D$ and $D \xrightarrow{G} C$ that are inverses of each other. However, this definition is in fact too restrictive. Here is a typical example why:

EXAMPLE 1.3.1. Let D be a category of finite-dimensional k-vector spaces and let C be its subcategory that has one object for each dimension n, namely the standard vector space k^n of column vectors.

Notice that $\mathbf{Mor}(k^n, k^m)$ can be identified with matrices $\mathrm{Mat}_{m,n}$ in the usual way of linear algebra. The categories C and D are not isomorphic, because D contains all sorts of vector spaces in each dimension, and C contains just one k^n. However, the main point of linear algebra is that C is somehow enough to do any calculation, because any n-dimensional vector space V is isomorphic to k^n "after we choose a basis in V".

Should we consider C and D as equivalent categories? To formalize this, we give the following definition:

DEFINITION 1.3.2. A covariant functor $C \xrightarrow{F} D$ is called an *equivalence of categories* if

- F is essentially surjective, i.e. any object in D is isomorphic (but not necessarily equal!) to an object of the form $F(X)$ for some $X \in C$.
- F is fully faithful, i.e.
$$\mathbf{Mor}_C(X, Y) = \mathbf{Mor}_D(F(X), F(Y))$$
for any objects $X, Y \in C$.

For example, let's return to "linear-algebra" categories above. We have an obvious inclusion functor $F : C \to D$. We claim that F is an equivalence of categories. To show that F is essentially surjective, take $V \in D$, i.e. V is an n-dimensional vector space. Then V is isomorphic to k^n, indeed any choice of a basis $e_1, \ldots, e_n \in V$ gives an isomorphism $V \to k^n$ which sends $v \in V$ to the column vector of its coordinates in the basis $\{e_i\}$. (an act of choice stipulates that we allow the axiom of choice, but let's not worry about such things). This shows that F is essentially surjective. Notice that F is fully faithful by definition: linear maps from k^n to k^m are the same in categories C and D. So F is an equivalence of categories.

Our definition has a serious flaw: it is not clear that equivalence of categories is an equivalence relation! We postpone the general statement to exercises, and here just look at our example: is there an equivalence of categories from D to C? We need a functor G from D to C. For any n-dimensional vector space V, there is only one candidate for $G(V)$: it must be k^n. Are we done? No, because we also have to define $G(L)$ for any linear map $L : V \to U$. So essentially, we need a matrix of L. This shows that there is no canonical choice for G: unlike F, G is not unique. However, we can do the following: let's choose a basis in each vector space V. In other words, let's choose a linear isomorphism $I_V : V \to k^n$ for each n-dimensional vector space V. Then we can define $G(L) : k^n \to k^m$ as the composition
$$k^n \xrightarrow{I_V^{-1}} V \xrightarrow{G} U \xrightarrow{I_U} k^m.$$
In more down-to-earth terms, $G(L)$ is a matrix of L in coordinates associated to our choice of bases in V and in U. Then it is immediate that G is essentially surjective (in fact just surjective) and it is easy to see that G is fully faithful: linear maps from V to U are identified with linear maps from k^n to k^m.

§1.4. **Representable Functors.** Fix an object $X \in C$. A very general and useful idea is to study X by poking it with other objects of C or by poking other objects by X. This is formalized as follows:

DEFINITION 1.4.1. A *contravariant* functor represented by X is a functor

$$h_X : C \to \mathbf{Sets}$$

that sends any $Y \in C$ to the set of morphisms $\mathbf{Mor}(Y, X)$ and that sends any morphism $Y_1 \xrightarrow{f} Y_2$ the function $\mathbf{Mor}(Y_2, X) \to \mathbf{Mor}(Y_1, X)$ obtained by taking composition with f.

Similarly, a *covariant* functor represented by X is a functor

$$h'_X : C \to \mathbf{Sets}$$

that sends any $Y \in C$ to the set of morphisms from X to $Y$and that sends any morphism $Y_1 \xrightarrow{f} Y_2$ the function $\mathbf{Mor}(X, Y_1) \to \mathbf{Mor}(X, Y_2)$ obtained by taking composition with f.

An interesting game is to start with a functor and try to guess if it's represented or not. For example, let's consider a forgetful covariant functor

$$\mathbf{Ab} \to \mathbf{Sets}$$

that sends any Abelian group to the set of its elements. Is it representable? We have to decide if there exists an Abelian group X such that morphisms from X to Y are in bijective correspondence with elements of Y. We claim that $X = \mathbb{Z}$ works. Indeed, a morphism from $\mathbb{Z}$ to an Abelian group Y is uniquely determined by the image of $1 \in \mathbb{Z}$. And for any element of Y, we can define a homomorphism $\mathbb{Z} \to Y$ that sends 1 to this element! So, quite remarkably, $h_{\mathbb{Z}}$ is nothing but the forgetful functor $\mathbf{Ab} \to \mathbf{Sets}$.

See exercises and Section §2.3 for further discussion and examples.

§1.5. **Products and Coproducts.** In some categories, such as $\mathbf{Sets}$ or $\mathbf{Vect}_k$, there is a natural notion of a product, for example if X and Y are two sets then $X \times Y$ is their Cartesian product. What could a definition of a product look like in other categories? If objects of our category are sets with some extra structure then we can try to define the product of two objects as their set-theoretic product endowed with this extra structure. For example, the product of two vector spaces U and V as a set is just the Cartesian product. Extra structures here are addition of vectors and multiplication of scalars: those are defined component-wise. But this approach clearly depends on the specific nature of the category at hand. And more importantly, it does not always work even in some very basic examples (such as fibered products of manifolds). Quite remarkably, there is another approach to products that does not use specifics of the category. Instead, it is based on the analysis of what the morphism from (or to) the product should look like. One can use the language of representable functors for this, but it will be easier to give an ad hoc definition.

DEFINITION 1.5.1. Let X and Y be objects of a category C. Their product (if it exists) is an object Z of C and two morphisms, $\pi_X : Z \to X$ and $\pi_Y : Z \to Y$ (called projections) such that the following "universal property" is satisfied. If W is another object of C endowed with morphisms $a : W \to X$

and $b : W \to Y$ then there exists a unique morphism $f : W \to Z$ such that $a = \pi_X \circ f$ and $b = \pi_Y \circ f$.

For example, suppose that X and Y are sets. Then we can take the Cartesian product $X \times Y$ as Z. The projections are just the usual projections: $\pi_X(x, y) = x$ and $\pi_Y(x, y) = y$. If we have functions $a : W \to X$ and $b : W \to Y$ then there is only one choice for a function $f : W \to X \to Y$, namely $f(w) = (a(w), b(w))$. So $X \times Y$ is indeed a product of X and Y according to the definition above.

A little tinkering with this definition gives coproducts:

DEFINITION 1.5.2. Let X and Y be objects of a category C. Their coproduct (if it exists) is an object Z of C and two morphisms, $i_X : X \to Z$ and $\pi_Y : Y \to Z$ such that the following "universal property" is satisfied. If W is another object of C endowed with morphisms $a : X \to W$ and $b : Y \to W$ then there exists a unique morphism $f : Z \to W$ such that $a = f \circ i_X$ and $b = f \circ i_Y$.

EXAMPLE 1.5.3. What is a coproduct of two sets? We claim that it is nothing but their disjoint union $X \sqcup Y$ with two inclusions $i_X : X \to X \sqcup Y$ and $i_Y : Y \to X \sqcup Y$. If we have maps $a : X \to W$ and $b : Y \to W$ then it is easy to define $f : X \sqcup Y \to W$: if $x \in X$ then $f(x) = a(x)$ and if $y \in Y$ then $f(y) = b(y)$.

EXAMPLE 1.5.4. What is a coproduct of two vector spaces, U and V? Taking the disjoint union of U and V is not a vector space in any reasonable way, so this is not the right way to go. It is quite remarkable that a coproduct exists, and is in fact equal to the product $U \times V$. The maps i_U and i_V are defined as follows: $i_U(u) = (u, 0)$ and $i_V(v) = (0, v)$. If we have maps $a : U \to W$ and $b : V \to W$ then $f : U \times V \to W$ is defined as follows: $f(u, v) = a(u) + b(v)$. It is quite easy to check that this is indeed a coproduct.

The difference between the product and coproduct of vector spaces becomes more transparent if we try to multiply more than two vector spaces. In fact, the product of any collection $\{V_i\}_{i \in I}$ of vector spaces is simply their Cartesian product (with a component-wise addition) but for a coproduct we have to make some changes, otherwise in the definition of the map f as in the previous Example we would have to allow infinite sums, which is not possible. In fact, the right definition of a coproduct is to take a direct sum $\bigoplus_{i \in I} V_i$. By definition, this is a subset of the direct product $\prod_{i \in I} V_i$ that parametrizes all collections $(v_i)_{i \in I}$ of vectors such that all but finitely many of v_i's are equal to 0. Then we can define the map f exactly as in the previous Example: if we have maps $a_i : V_i \to W$ for any i then $f : \bigoplus_{i \in I} V_i \to W$ takes $(v_i)_{i \in I}$ to $\sum_i a_i(v_i)$.

§1.6. **Natural Transformations.** As Saunders Maclane famously said: "I did not invent category theory to talk about functors. I invented it to talk about natural transformations." So what is a natural transformation? It is a map form one functor to another! Let me start with an example that explains why we might need such a thing.

Recall that for any vector space V, we have a "natural" linear map

$$\alpha_V : V \to V^{**}$$

(in fact an isomorphism if $\dim V < \infty$) that sends a vector $v \in V$ to the linear functional $f \mapsto f(v)$ on V^*. How is this map "natural"?

One explanation is that α_V does not depend on any choices. After all, if $\dim V < \infty$ then V and V^* are isomorphic as well but there is no special choice for this isomorphism unless we fix a basis of V. But this explanation is still "linguistic", the question is, can we *define* naturality mathematically?

To get to the answer, let's study the effect of α_V on morphisms (this is a general recipe of category theory, look not just at objects but also at morphisms). Let $U \xrightarrow{L} V$ be a linear map. We also have our "natural" linear maps $\alpha_U : U \to U^{**}$ and $\alpha_V : V \to V^{**}$. Finally, by taking a contragredient linear map twice, we have a contragredient linear map $U^{**} \xrightarrow{L^{**}} V^{**}$. To summarize things, we have a square of linear maps:

$$\begin{array}{ccc} U & \xrightarrow{\alpha_U} & U^{**} \\ L\big\downarrow & & \big\downarrow L^{**} \\ V & \xrightarrow[\alpha_V]{} & V^{**} \end{array} \tag{1}$$

Apriori, there is no reason for this diagram to be commutative: if α_U were a random linear map, this diagram obviously won't be commutative. However, it is easy to see that this diagram is commutative. Let's show it by chasing the diagram. Pick $u \in U$. Then we claim that

$$\alpha_V(L(u)) = L^{**}(\alpha_U(u)).$$

Both sides of this equation are elements of V^{**}, i.e. linear functionals on V^*. The functional on the LHS takes $f \in V^*$ to $f(L(u))$. The functional on the RHS takes $f \in V^*$ to

$$\alpha_U(u)(L^*(f)) = L^*(f)(u) = f(L(u)).$$

This calculation might look confusing, but I don't think there is any way to make it more palatable, my only suggestion is to redo this calculation yourself!

Now let's give a general definition.

DEFINITION 1.6.1. Let $F, G : C_1 \to C_2$ be two covariant functors. A *natural transformation* $\alpha : F \to G$ between them is a rule that, for each object $X \in C_1$, assigns a morphism $F(X) \xrightarrow{\alpha_X} G(X)$ in C_2 such that the following condition is satisfied. For any morphism $X_1 \xrightarrow{f} X_2$ in C_1, we have a commutative diagram

$$\begin{array}{ccc} F(X_1) & \xrightarrow{\alpha_{X_1}} & G(X_1) \\ F(f)\big\downarrow & & \big\downarrow G(f) \\ F(X_2) & \xrightarrow[\alpha_{X_2}]{} & G(X_2) \end{array} \tag{2}$$

If α_X is an isomorphism for any X then α is called a *natural isomorphism*.

How is this related to the linear algebra example above? Let $\mathbf{Vect}_k$ be the category of vector spaces over k. Consider two functors: the identity functor $\mathrm{Id} : \mathbf{Vect}_k \to \mathbf{Vect}_k$ and the "double duality" functor $D : \mathbf{Vect}_k \to \mathbf{Vect}_k$ that sends any vector space V to V^{**} and any linear map $L : U \to V$ to a double contragredient linear map $L^{**} : U^{**} \to V^{**}$.

We claim that there is a natural transformation from Id to D (and in fact a natural isomorphism if we restrict to a subcategory of finite-dimensinal vector spaces). All we need is a rule α_V for each vector space: it should be a morphism, i.e. a linear map, from $\mathrm{Id}(V) = V$ to $D(V) = V^{**}$ such that (2) is satisfied for any morphism $U \to V$. This is exactly the linear map we have constructed above, and (1) is a commutative square we need.

See exercises and Section §2.3 for further discussion and examples.

§1.7. **Exercises.**

1. Let C be a category. (a) Prove that an identity morphism $A \to A$ is unique for each object $A \in \mathbf{Ob}(C)$. (b) Prove that each isomorphism in C has a unique inverse.

2. Let C be a category. An object X of C is called an *initial* object (resp. a *terminal* object) if, for every object Y of C, there exists a unique morphism $X \to Y$ (resp. a unique morphism $Y \to X$). (a) Decide if the following categories contain initial objects, and if so, describe them: the category of vector spaces, the category of groups, the category of commutative rings (with 1). (b) Prove that a terminal object (if exists) is unique up to a canonical isomorphism (and what exactly does it mean?).

3. Let $(I, \leq)$ be a poset (partially ordered set) and let C_I be the corresponding category. Unwind definitions (i.e. give definitions in terms of the poset, without using any categorical language) of (a) terminal and initial objects in C_I (if they exist); (b) product and coproduct in C_I (if they exist).

4. Let X be a fixed object of a category C. We define a new category C/X of *objects of C over X* as follows: an object of C/X is an object Y of C along with some morphism $Y \to X$. In other words, an object of C/X is an arrow $Y \to X$. A morphism from $Y \to X$ to $Y' \to X$ is a morphism from Y to Y' that makes an obvious triangle commutative. Prove that C/X is indeed a category and that $1_X : X \to X$ is its terminal object.

5. In the notation of Problem 3, let C_I be the category associated with a poset I and let $\mathbf{Ab}$ be the category of Abelian groups. A contravariant functor $C_I \to \mathbf{Ab}$ is called an *inverse system* of Abelian groups indexed by a partially ordered set I. (a) Reformulate this definition without using categorical language. (b) Consider Abelian groups $\mathbb{Z}/2^n\mathbb{Z}$ for $n = 1, 2, \ldots$ and natural homomorphisms $\mathbb{Z}/2^n\mathbb{Z} \to \mathbb{Z}/2^m\mathbb{Z}$ for $n \geq m$. Show that this is an inverse system. (c) Let C be an arbitrary category. Give a definition of an inverse system of objects in C indexed by a poset I. Show that (b) is an inverse system of rings.

6. In the notation of Problem 4, fix some inverse system $F : C_I \to \mathbf{Ab}$. Also, let's fix an Abelian group A and consider an inverse system $F_A : C_I \to \mathbf{Ab}$ defined as follows: $F_A(i) = A$ for any $i \in I$ and if $i \leq j$ then the corresponding morphism $A \to A$ is the identity. (a) Prove that F_A is indeed an inverse system. (b) Show that the rule $A \to F_A$ can be extended to a functor from the category $\mathbf{Ab}$ to the category of inverse systems $C_I \to \mathbf{Ab}$ (with natural transformations as morphisms). (c) Unwind definitions to describe what it means to have a natural transformation from F_A to F without categorical language.

7. In the notation of Problem 6, an Abelian group A is called an *inverse limit* of an inverse system $F : C_I \to \mathbf{Ab}$ if for any Abelian group B, and for any natural transformation $F_B \to F$, there exists a unique homomorphism $B \to A$ such that F_B factors through F_A. (a) Unwind definitions to describe the inverse limit without categorical language. (b) Show that the inverse system of rings in Problem 5(b) has an inverse limit (called the ring of 2-adic numbers).

8. Let $F : \mathbf{Sets} \to \mathbf{Sets}$ be a contravariant functor that sends any set S to the set of subsets of S and any function $f : S \to S'$ to a function that sends $U \subset S'$ to $f^{-1}(U) \subset S$. (a) Show that F is representable by a two-element set $\{0, 1\}$. (b) Describe a contravariant functor representable by a three-element set $\{0, 1, 2\}$.

9. Let V be a real vector space. Prove that its complexification $V_{\mathbb{C}}$ represents the covariant functor $F : \mathbf{Vect}_{\mathbb{C}} \to \mathbf{Sets}$. Namely, for any complex vector space U, $F(U)$ is the set of $\mathbb{R}$-linear maps $V \to U_{\mathbb{R}}$ (where $U_{\mathbb{R}}$ is U considered as a real vector space).

10. Let C and D be categories and let $F : C \to D$ and $G : D \to C$ be functors. Then F is called a *left adjoint* of G (and G is called a *right adjoint* of F) if, for each pair of objects $X \in C$ and $Y \in D$, there exist bijections of sets

$$\tau_{X,Y} : \mathbf{Mor}_D(F(X), Y) \to \mathbf{Mor}_C(X, G(Y))$$

that are natural transformations in X for fixed Y and in Y for fixed X. (a) Explain what this last condition means explicitly. (b) Show that complexification $\mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{C}}$ and restriction of scalars $\mathbf{Vect}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{R}}$ are adjoint functors.

11. Let $G : \mathbf{Vect}_k \to \mathbf{Sets}$ be a forgetful functor. Describe its left-adjoint.

12. Let C be a category and let $X, Y \in \mathbf{Ob}(C)$. Consider representable functors $C \to \mathbf{Sets}$ given by X and Y, i.e. $h_X = \mathbf{Mor}(\cdot, X)$ and $h_Y = \mathbf{Mor}(\cdot, Y)$. Show that there is a natural bijection between morphisms $X \to Y$ and natural transformations $h_X \to h_Y$. More precisely, let D be a category of functors $C \to \mathbf{Sets}$ (with natural transformations as morphisms). Show that the rule $X \to h_X$ extends to a fully-faithful functor $C \to D$.

13. Show that equivalence of categories is an equivalence relation on categories, i.e. if C and D are equivalent then D and C are also equivalent, and that if C and D (resp. D and E) are equivalent then C and E are also equivalent. This relations is obviously reflexive: any category is equivalent to itself by means of the identity functor $\mathrm{Id}_C : C \to C$.

14. Give example of a category where (a) products do not always exist; (b) products exist but coproducts do not always exist.

§2. Tensor Products

§2.1. **Tensor Product of Vector Spaces.** Let's define tensor products in the category of vector spaces over a field k. Fix two vector spaces, U and V.

We want to understand all bilinear maps

$$U \times V \xrightarrow{\beta} W,$$

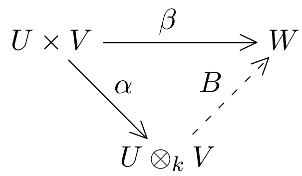
where W can be any vector space. For example, if $W = k$, then β is just a bilinear function. We are not going to fix W, instead we allow it to vary.

Notice that if $U \times V \to \tilde{W}$ is a bilinear map, and $\tilde{W} \to W$ is a linear map, then the composition $U \times V \to \tilde{W} \to W$ is again bilinear. So we can ask if there exists the "biggest" bilinear map $U \times V \to \tilde{W}$ such that any other bilinear map $U \times V \to W$ factors through some linear map $\tilde{W} \to W$. It turns out that this universal $\tilde{W}$ exists. It is known as a tensor product.

DEFINITION 2.1.1. A vector space $U \otimes_k V$, and a bilinear map

$$U \times V \xrightarrow{\alpha} U \otimes_k V$$

is called a *tensor product* if, for any bilinear map $U \times V \xrightarrow{\beta} W$, there exists a unique linear map $U \otimes_k V \xrightarrow{B} W$ (called a *linear extension* of β) such that the following diagram commutes:



(3)

THEOREM 2.1.2. *The tensor product exists and is unique (up to isomorphism).*

We will prove this theorem later, when we discuss more general tensor products of R-modules. But first let's analyze how $U \otimes_k V$ looks like.

DEFINITION 2.1.3. For any pair $(u, v) \in U \times V$, its image $\alpha(u, v) \in U \otimes_k V$ is called a *pure tensor* or an *indecomposable tensor*, and it is denoted by $u \otimes v$.

LEMMA 2.1.4. *$U \otimes_k V$ is spanned by pure tensors (but be careful, not any element of $U \otimes_k V$ is a pure tensor!) We have bilinear relations between pure tensors:*

$$(au_1 + bu_2) \otimes v = a(u_1 \otimes v) + b(u_2 \otimes v), \tag{4}$$

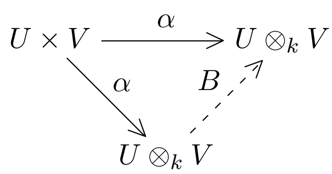
$$u \otimes (av_1 + bv_2) = a(u \otimes v_1) + b(u \otimes v_2). \tag{5}$$

If $\{e_i\}$ is a basis of U and $\{f_j\}$ is a basis of V then $\{e_i \otimes f_j\}$ is a basis of $U \otimes_k V$. In particular,

$$\dim(U \otimes_k V) = (\dim U) \cdot (\dim V)$$

(assuming U and V are finite-dimensional).

Proof. We are going to define various interesting bilinear maps and analyze the universal property (3). For example, let's take a bilinear map $\beta = \alpha$:



Commutativity of the diagram simply means that

$$B(u \otimes v) = \alpha(u, v) = u \otimes v$$

for any pair (u, v). So we see that the restriction of B to the linear span of pure tensors must be the identity map. Suppose that pure tensors don't span the whole $U \otimes_k V$. Then there are many ways to extend a linear map B from the linear span of pure tensors to the whole $U \otimes_k V$. In particular, B is not unique, which contradicts the universal property.

The fact that pure tensors satisfy bilinear relations simply follows from the fact that α is a bilinear map. For example,

$$\alpha(au_1 + bu_2, v) = a\alpha(u_1, v) + b\alpha(u_2, v),$$

which by definition implies

$$(au_1 + bu_2) \otimes v = a(u_1 \otimes v) + b(u_2 \otimes v).$$

It follows from bilinearity that if $u = \sum x_i e_i$ and $v = \sum y_j f_j$ then

$$u \otimes v = \sum x_i y_j (e_i \otimes f_j).$$

Since $U \otimes_k V$ is spanned by pure tensors, we see that in fact $U \otimes_k V$ is spanned by vectors $e_i \otimes f_j$. To show that these vectors form a basis, it remains to show that they are linearly independent.

Suppose that some linear combination is trivial:

$$\sum a_{ij} e_i \otimes f_j = 0. \tag{6}$$

How to show that each $a_{ij} = 0$? Let's fix two indices, i_0 and j_0, and consider a bilinear function $U \times V \xrightarrow{\beta} k$ defined as follows:

$$\beta\left(\sum x_i e_i, \sum y_j f_j\right) = x_{i_0} y_{j_0}.$$

Then $\beta(e_{i_0}, f_{j_0}) = 1$ and $\beta(e_i, f_j) = 0$ for any other pair of basis vectors. Now we compute its linear extension applied to our linear combination:

$$B\left(\sum a_{ij} e_i \otimes f_j\right) = \sum a_{ij} B(e_i \otimes f_j) = a_{i_0 j_0}.$$

On the other hand,

$$B\left(\sum a_{ij} e_i \otimes f_j\right) = B(0) = 0.$$

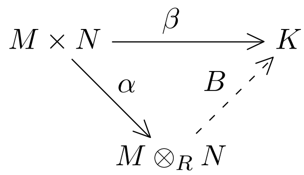
So all coefficients a_{ij} in (6) must vanish. □

§2.2. **Tensor Product of R-modules.** We will extend the notion of tensor products to the category $\mathbf{Mod}_R$ of R-modules, where R is a commutative ring with 1. To stress analogy with vector spaces, instead of saying "homomorphism of R-modules", we will say "R-linear map of R-modules". We fix two R-modules, M and N and study R-bilinear maps $M \times N \to K$, where K is an arbitrary R-module. The definition and the main theorem are the same:

DEFINITION 2.2.1. An R-module $M \otimes_R N$ endowed with an R-bilinear map

$$M \times N \xrightarrow{\alpha} M \otimes_R N$$

is called a *tensor product* if, for any R-bilinear map $M \times N \xrightarrow{\beta} K$, there exists a unique R-linear map $M \otimes_R N \xrightarrow{B} K$ (called a *linear extension* of β) such that the following diagram commutes:



(7)

THEOREM 2.2.2. *The tensor product exists.*

Proof. We are just going to define $M \otimes_R N$ as an R-module generated by pure tensors $u \otimes v$ modulo bilinear relations (4) and (5). But to avoid notational chaos, let's proceed a bit more formally. Let W be a free R-module with one basis vector $[m, n]$ for each pair of elements $m \in M$, $n \in N$. There are many pairs, so this is a really huge R-odule! Let $W_0 \subset W$ be a submodule spanned by all expressions

$$[au_1 + bu_2, v] - a[u_1, v] - b[u_2, v]$$

and

$$[u, av_1 + bv_2] - a[u, v_1] - b[u, v_2].$$

We define

$$M \otimes_R N := W/W_0$$

(quotient R-module). We define pure tensors $u \otimes v$ as cosets of $[u, v]$:

$$u \otimes v := [u, v] + W_0.$$

Then we have

$$(au_1 + bu_2) \otimes v = a(u_1 \otimes v) + b(u_2 \otimes v)$$

and

$$u \otimes (av_1 + bv_2) = a(u \otimes v_1) + b(u \otimes v_2).$$

We define a map

$$M \times N \xrightarrow{\alpha} M \otimes_R N, \quad \alpha(u, v) = u \otimes v.$$

Equations above show that α is bilinear.

Why does α satisfy the universal property (7)? Given a bilinear map $M \times N \xrightarrow{\beta} K$, we can define an R-linear map $W \xrightarrow{f} K$ by a simple rule

$$f([u, v]) = \beta(u, v)$$

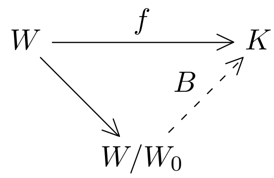
(notice that an R-linear map from a free R-module can be defined, and is uniquely determined, by its values on the basis). We claim that $W_0 \subset \operatorname{Ker} f$. It is enough to check that f kills generators of f. And indeed, we have

$$f\big([au_1 + bu_2, v] - a[u_1, v] - b[u_2, v]\big) = \beta(au_1 + bu_2, v) - a\beta(u_1, v) - \beta(u_2, v) = 0$$

and

$$f\big([u, av_1 + bv_2] - a[u, v_1] - b[u, v_2]\big) = \beta(u, av_1 + bv_2) - a\beta(u, v_1) - b\beta(u, v_2) = 0$$

by bilinearity of β. It follows that f defines an R-linear map $W/W_0 \xrightarrow{B} K$:



This map is our bilinear extension $B : M \otimes_R N \to K$.

Finally, notice that we have no choice but to define

$$B(u \otimes v) = \beta(u, v)$$

if we want the diagram (7) to be commutative. So B is unique and $M \otimes_R N$ indeed satisfies the universal property of the tensor product. $\square$

We can generalize Lemma 2.1.4:

LEMMA 2.2.3. *$M \otimes_R N$ is spanned by pure tensors. If M is a free R-module with basis $\{e_i\}$ and N is a free R-module with basis $\{f_j\}$ then $M \otimes N$ is a free R-module with basis $\{e_i \otimes f_j\}$.*

Proof. The proof is identical to the proof of Lemma 2.1.4. $\square$

EXAMPLE 2.2.4. Tensor products of non-free R-modules are much more interesting. For example, suppose that $R = \mathbb{Z}$, i.e. we are computing tensor products of Abelian groups. What is $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3$? Consider a pure tensor $a \otimes b \in \mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3$. Since $a = 3a$ in $\mathbb{Z}_2$, we have

$$a \otimes b = (3a) \otimes b = 3(a \otimes b) = a \otimes (3b) = a \otimes 0 = 0.$$

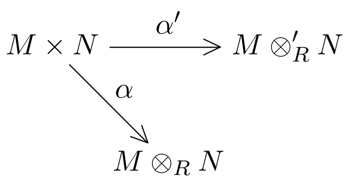
Since $\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3$ is spanned by pure tensors, we have

$$\mathbb{Z}_2 \otimes_{\mathbb{Z}} \mathbb{Z}_3 = 0.$$

Next we discuss uniqueness of tensor products.

THEOREM 2.2.5. *Tensor product $M \otimes_R N$ is unique up to a canonical isomorphism.*

Proof. Suppose that we have two R-modules, let's call them $M \otimes_R N$ and $M \otimes'_R N$, and two bilinear maps, $M \times N \xrightarrow{\alpha} M \otimes_R N$ and $M \times N \xrightarrow{\alpha'} M \otimes'_R N$ that both of them satisfy the universal property. From the diagram



(8)

we deduce existence of unique linear maps

$$M \otimes_R N \xrightarrow{B} M \otimes'_R N \quad \text{and} \quad M \otimes'_R N \xrightarrow{B'} M \otimes_R N$$

that make (8) commutative. We claim that B is an isomorphism and B' is its inverse. Indeed, $B' \circ B$ makes the following diagram commutative:

$$\begin{array}{ccc} M \times N & \xrightarrow{\alpha} & M \otimes_R N \\ & {\scriptstyle \alpha} \searrow \quad \nearrow {\scriptstyle B' \circ B} & \\ & M \otimes_R N & \end{array}$$

But the identity map on $M \otimes_R N$ also makes it commutative. By uniqueness of the linear extension, we see that $B' \circ B = \mathrm{Id}\,|_{M \otimes_R N}$. A similar argument shows that $B \circ B' = \mathrm{Id}\,|_{M \otimes'_R N}$. □

This argument shows that if we have two R-modules that satisfy the universal property of the tensor product, then they are not only isomorphic, but in fact there is a canonical *choice* for this isomorphism (given by maps B and B' of the proof). That's why we say that the tensor product $M \otimes_R N$ is unique up to a *canonical* isomorphism. The argument used in the proof above is very general. It can be easily generalized if we recast it in the categorical language. This is done in the next section.

§2.3. **Categorical aspects of tensors: Yoneda's Lemma.**

DEFINITION 2.3.1. Fix R-modules M and N and define a covariant functor

$$\mathit{BilMaps}_{M,N} : \mathbf{Mod}_R \to \mathbf{Sets}$$

that sends any R-module K to the set of bilinear maps

$$\{\beta \,|\, M \times N \xrightarrow{\beta} K\}$$

and that sends any R-linear map $K \xrightarrow{f} K'$ to the function

$$\{\beta \,|\, M \times N \xrightarrow{\beta} K\} \to \{\beta' \,|\, M \times N \xrightarrow{\beta'} K'\}$$

that assigns to a bilinear function $M \times N \xrightarrow{\beta} K$ with values in K a bilinear function $M \times N \xrightarrow{\beta} K \xrightarrow{f} K'$ with values in K'.

The R-module $M \otimes_R N$, as any other R-module, defines a covariant *representable functor*

$$h_{M \otimes_R N} : \mathbf{Mod}_R \to \mathbf{Sets}$$

that sends an R-module K to the set of R-linear maps

$$\{B \,|\, M \otimes_R N \xrightarrow{B} K\}$$

and that sends an R-linear map $K \xrightarrow{f} K'$ to the function

$$\{B \,|\, M \otimes_R N \xrightarrow{B} K\} \to \{B' \,|\, M \otimes_R N \xrightarrow{B'} K'\}$$

that assigns to an R-linear function $M \otimes_R N \xrightarrow{B} K$ with values in K an R-function $M \otimes_R N \xrightarrow{B} K \xrightarrow{f} K'$ with values in K'.

Now of course the whole point of introducing the tensor product is to identify the set of bilinear maps $M \times N \to K$ with the set of linear maps

$M \otimes_R N \to K$. How exactly is this done? Recall that we also have a "universal" bilinear map

$$M \times N \xrightarrow{\alpha} M \otimes_R N.$$

For any linear map $M \otimes_R N \xrightarrow{B} K$, $B \circ \alpha$ is a bilinear map $M \times N \longrightarrow K$. And vice versa, for any bilinear map $M \times N \xrightarrow{\beta} K$, there exists a unique linear map $M \otimes_R N \xrightarrow{B} K$ such that $B \circ \alpha = \beta$.

In other words, for any R-module K, we have a bijection of sets

$$h_{M \otimes_R N}(K) \xrightarrow{\alpha_K} \mathit{BilMaps}_{M,N}(K)$$

where α_K simply composes a linear map $M \otimes_R N \to K$ with α.

LEMMA 2.3.2. *This gives a natural isomorphism of functors*

$$\alpha : h_{M \otimes_R N} \to \mathit{BilMaps}_{M,N}.$$

Proof. Natural transformations and natural isomorphisms are defined in Section §1.6. We need a rule that for each R-module K gives a bijection α_K of sets (recall that isomorphisms in the category of sets are called bijections)

$$h_{M \otimes_R N}(K) \to \mathit{BilMaps}_{M,N}(K)$$

from the set of linear maps $M \otimes_R N \to K$ to the set of bilinear maps $M \times N \to K$. We have already defined this bijection, this is just a bijection given by taking composition with a universal bilinear map $M \times N \to M \otimes_R N$.

It remains to check that the square (2) is commutative. Take an R-linear map $K_1 \xrightarrow{f} K_2$. We have to check that the following square is commutative:

$$\begin{array}{ccc} h_{M \otimes_R N}(K_1) & \xrightarrow{\alpha_{K_1}} & \mathit{BilMaps}_{M,N}(K_1) \\ {\scriptstyle h_{M \otimes_R N}(f)} \downarrow & & \downarrow {\scriptstyle \mathit{BilMaps}_{M,N}(f)} \\ h_{M \otimes_R N}(K_2) & \xrightarrow[\alpha_{K_2}]{} & \mathit{BilMaps}_{M,N}(K_2) \end{array}$$

Let's chase the diagram. Take an element of $h_{M \otimes_R N}(K_1)$, i.e. an R-linear map

$$M \otimes_R N \xrightarrow{B} K_1.$$

The horizontal arrow α_{K_1} takes it to the bilinear map

$$M \times N \xrightarrow{\alpha} M \otimes_R N \xrightarrow{B} K_1$$

and then the vertical map $\mathit{BilMaps}_{M,N}(f)$ takes it to the bilinear map

$$M \times N \xrightarrow{\alpha} M \otimes_R N \xrightarrow{B} K_1 \xrightarrow{f} K_2.$$

On the other hand, if we apply the vertical arrow $h_{M \otimes_R N}(f)$ first, we will get a linear map

$$M \otimes_R N \xrightarrow{B} K_1 \xrightarrow{f} K_2$$

and applying α_{K_2} gives a bilinear map

$$M \times N \xrightarrow{\alpha} M \otimes_R N \xrightarrow{B} K_1 \xrightarrow{f} K_2,$$

the same as above. So the square commutes. □

If we can define a tensor product of M and N in two different ways, say $M \otimes_R N$ and $M \otimes'_R N$, the representable functors $h_{M\otimes_R N}$ and $h_{M\otimes'_R N}$ are going to be naturally isomorphic (because both of them are naturally isomorphic to $BilMaps_{M,N}$). So to reprove Theorem 2.2.5, we can use the following weak version of Yoneda's lemma:

LEMMA 2.3.3. *Let X, Y be two objects in a category C. Suppose we have a natural isomorphism of representable functors $\alpha : h_X \to h_Y$. Then X and Y are canonically isomorphic.*

Proof. To match our discussion of the tensor product, we will prove a covariant version, the contravariant version has a similar proof. Recall that h_X sends any object Z to the set $\mathbf{Mor}(X, Z)$ and it sends any morphism $Z_1 \to Z_2$ to the function $\mathbf{Mor}(X, Z_1) \to \mathbf{Mor}(X, Z_2)$ obtained by taking a composition with $Z_1 \to Z_2$.

So α gives, for any object Z in C, a bijection

$$\alpha_Z : \mathbf{Mor}(X, Z) \to \mathbf{Mor}(Y, Z)$$

such that for each morphism $Z_1 \to Z_2$ we have a commutative diagram

$$\begin{array}{ccc} \mathbf{Mor}(X, Z_1) & \xrightarrow{\alpha_{Z_1}} & \mathbf{Mor}(Y, Z_1) \\ \downarrow & & \downarrow \\ \mathbf{Mor}(X, Z_2) & \xrightarrow[\alpha_{Z_2}]{} & \mathbf{Mor}(Y, Z_2) \end{array}$$

where the vertical arrows are obtained by composing with $Z_1 \to Z_2$.

In particular, we have bijections

$$\mathbf{Mor}(X, X) \xrightarrow{\alpha_X} \mathbf{Mor}(Y, X) \quad \text{and} \quad \mathbf{Mor}(X, Y) \xrightarrow{\alpha_Y} \mathbf{Mor}(Y, Y).$$

We define morphisms

$$f = \alpha_X(\mathrm{Id}_X) \in \mathbf{Mor}(Y, X) \quad \text{and} \quad g = \alpha_Y^{-1}(\mathrm{Id}_Y) \in \mathbf{Mor}(X, Y).$$

We claim that f and g are inverses of each other, and in particular X and Y are canonically isomorphic (by f and g). Indeed, consider the commutative square above when $Z_1 = X$, $Z_2 = Y$, and the morphism from X to Y is g. It gives

$$\begin{array}{ccc} \mathbf{Mor}(X, X) & \xrightarrow{\alpha_X} & \mathbf{Mor}(Y, X) \\ {\scriptstyle g\circ\cdot}\downarrow & & \downarrow{\scriptstyle g\circ\cdot} \\ \mathbf{Mor}(X, Y) & \xrightarrow[\alpha_Y]{} & \mathbf{Mor}(Y, Y) \end{array}$$

Let's take $\mathrm{Id}_X \in \mathbf{Mor}(X, X)$ and compute its image in $\mathbf{Mor}(Y, Y)$ in two different ways. If we go horizontally, we first get $\alpha_X(\mathrm{Id}_X) = f \in \mathbf{Mor}(Y, X)$. Then we take its composition with $X \xrightarrow{g} Y$ to get $g \circ f \in \mathbf{Mor}(Y, Y)$. If we go vertically first, we get $g \in \mathbf{Mor}(X, Y)$. Then we get $\alpha_Y(g) = \mathrm{Id}_Y$, because $g = \alpha_Y^{-1}(\mathrm{Id}_Y)$. So we see that $g \circ f = \mathrm{Id}_Y$. Similarly, one can show that $f \circ g = \mathrm{Id}_X$, i.e. f and g are really inverses of each other. □

The full (covariant) version of the Yoneda's lemma is this:

LEMMA 2.3.4. *Let C be a category. For any object X of C, consider a covariant representable functor $h_X : C \to \mathbf{Sets}$. For any morphism $X_1 \xrightarrow{f} X_2$, consider a natural transformation $h_{X_2} \to h_{X_1}$ defined as follows: for any object Y of C, the function*

$$\alpha_Y : h_{X_2}(Y) = \mathbf{Mor}(X_2, Y) \xrightarrow{\cdot \circ f} \mathbf{Mor}(X_1, Y) = h_{X_1}(Y)$$

is just a composition of $g \in \mathbf{Mor}(X_2, Y)$ with $X_1 \xrightarrow{f} X_2$. This gives a functor from C to the category of covariant functors $C \to \mathbf{Sets}$ (with natural transformations as morphisms).

This functor is fully faithful, i.e. the set of morphisms $X_1 \to X_2$ in C is identified with the set of natural transformations $h_{X_2} \to h_{X_1}$.

Proof. For any morphism $X_1 \xrightarrow{f} X_2$, the natural transformation $\alpha : h_{X_2} \to h_{X_1}$ is defined in the statement of the Lemma. Now suppose we are given a natural transformation $\alpha : h_{X_2} \to h_{X_1}$. Applying α_{X_2} to $\mathrm{Id}_{X_2} \in \mathbf{Mor}(X_2, X_2)$ gives some morphism $f \in \mathbf{Mor}(X_1, X_2)$. We claim that this establishes a bijection between $\mathbf{Mor}(X_1, X_2)$ and natural transformations $h_{X_2} \to h_{X_1}$.

Start with $f \in \mathbf{Mor}(X_1, X_2)$. Then $\alpha_{X_2} : \mathbf{Mor}(X_2, X_2) \to \mathbf{Mor}(X_1, X_2)$ is obtained by composing with f. In particular, $\alpha_{X_2}(\mathrm{Id}_{X_2}) = f$.

Finally, let us start with a natural transformation $\alpha : h_{X_2} \to h_{X_1}$. Then

$$f = \alpha_{X_2}(\mathrm{Id}_{X_2}) \in \mathbf{Mor}(X_1, X_2).$$

It defines a natural transformation $\beta : h_{X_2} \to h_{X_1}$. We have to show that $\alpha = \beta$, i.e. that for any $Y \in C$, the map $\alpha_Y : \mathbf{Mor}(X_2, Y) \to \mathbf{Mor}(X_1, Y)$ is just a composition with f. The argument is the same as in the previous Lemma. Start with any $g \in \mathbf{Mor}(X_2, Y)$ and consider a commutative square

$$\begin{array}{ccc}
\mathbf{Mor}(X_2, X_2) & \xrightarrow{\alpha_{X_2}} & \mathbf{Mor}(X_1, X_2) \\
\downarrow & & \downarrow \\
\mathbf{Mor}(X_2, Y) & \xrightarrow[\alpha_Y]{} & \mathbf{Mor}(X_1, Y)
\end{array}$$

where the vertical maps are compositions with $X_2 \xrightarrow{g} Y$. Take Id_{X_2} and follow it along the diagram. We get

$$\begin{array}{ccc}
\mathrm{Id}_{X_2} & \xrightarrow{\alpha_{X_2}} & f \\
\downarrow & & \downarrow \\
g & \xrightarrow[\alpha_Y]{} & \alpha_Y(g) = g \circ f
\end{array}$$

So $\alpha_Y(g)$ is exactly what we want: simply a composition with f. □

Why is Yoneda's lemma useful? Very often we have to deal with situations when it is hard to construct a morphism $X \to Y$ between two objects in the category directly. For example, it is hard to construct an explicit differentiable map from one manifold to another. Yoneda's lemma gives an indirect way of constructing morphisms. Of course, it works only if we have a good understanding of (covariant or contravariant) functors h_X and h_Y. In this case we can try to define a natural transformation between these functors instead of defining the morphism $X \to Y$ directly. Let's work out a simple example of this.

LEMMA 2.3.5. *For any R-modules M and N, we have a canonical isomorphism*

$$M \otimes_R N \simeq N \otimes_R M.$$

Proof. Of course this isomorphism just takes a pure tensor $m \otimes n$ to $n \otimes m$. But since pure tensors are linearly dependent, we have to check that this morphism is well-defined. For example, we can look at a bilinear map $M \times N \to N \otimes_R M$ that sends $(m, n) \to n \otimes m$ and use the universal property to factor this bilinear map through the tensor product $M \otimes_R N$.

Let's repackage this argument to highlight how Yoneda's lemma works. We already know that $h_{M \otimes_R N}$ is naturally isomorphic to the functor of bilinear maps $\mathit{BilMaps}_{M,N}$ and of course $h_{N \otimes_R M}$ is naturally isomorphic to the functor $\mathit{BilMaps}_{N,M}$. So, by Yoneda's lemma, to construct an explicit isomorphism between $M \otimes_R N$ and $N \otimes_R M$ it suffices to construct an explicit natural isomorphism between functors $\mathit{BilMaps}_{M,N}$ and $\mathit{BilMaps}_{N,M}$. In other words, for each R-module K, we need a bijection α_K between $\mathit{BilMaps}_{M,N}(K)$ and $\mathit{BilMaps}_{N,M}(K)$, i.e. between the set of bilinear maps $M \times N \to K$ and the set of bilinear maps $N \times M \to K$ that behaves "naturally" in K, i.e. for each R-linear map $K_1 \to K_2$, the following diagram commutes

$$\begin{array}{ccc} \mathit{BilMaps}_{M,N}(K_1) & \xrightarrow{\alpha_{K_1}} & \mathit{BilMaps}_{N,M}(K_1) \\ \downarrow & & \downarrow \\ \mathit{BilMaps}_{M,N}(K_2) & \xrightarrow{\alpha_{K_2}} & \mathit{BilMaps}_{N,M}(K_2) \end{array}$$

where the vertical maps are just compositions with $K_1 \to K_2$. It is clear that α_K is a very simple transformation: it just takes a bilinear map $M \times N \xrightarrow{\beta} K$ to a bilinear map

$$N \times M \to M \times N \xrightarrow{\beta} K,$$

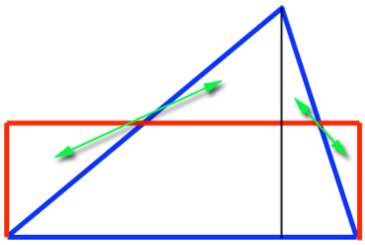
where the first map is a switch $(n, m) \to (m, n)$. □

§2.4. **Hilbert's 3d Problem.** As a fun application of tensors, let's solve the Hilbert's 3d problem:

PROBLEM 2.4.1. *Given two polytopes $P, Q \subset \mathbb{R}^3$ of the same volume, is it always possible to cut P into polyhedral pieces and then reassemble them into Q?*

Here a polytope is a 3-dimensional analogue of a polygon: we can define it, for example, as a convex hull of finitely many points in $\mathbb{R}^3$.

For polygons, i.e. in dimension 2, the problem above has a positive solution, which can be seen by applying induction and various simple cutting tricks. For example, it is easy to cut a triangle and then rearrange pieces to



get a rectangle: Notice that this actually *proves* that the area of a triangle is equal to $ah/2$, where a is the base and h is the height.

This is a source of many cute puzzles, for example Figure 1 shows how to cut a square into pieces that can be rearranged to get a regular hexagon.

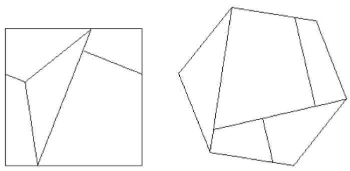


FIGURE 1. Cut and Paste

If the answer to the 3-dimensional Problem were positive, it would be possible to derive volume formulas for polytopes using geometry only. However, it was known since Archimedes that to prove the volume formula even for a tetrahedron, one has to integrate! So people have long suspected (at least since Gauss) that the answer to the Problem is negative.

After Hilbert stated his famous problems, the third problem was almost immediately solved by his student, Max Dehn. Dehn's idea was to introduce some sort of a hidden volume: some invariant of polytopes different from volume that nevertheless behaves additively if you cut a polytope into pieces. To be more specific, let

$$\Gamma = \mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{R}/\pi\mathbb{Z})$$

(the tensor product of Abelian groups).

DEFINITION 2.4.2. For a polytope P, let $E_1, \ldots, E_r$ be the collection of its edges. For each edge E_i, let l_i be its length and let α_i be the angle between faces meeting along E_i. We define the *Dehn invariant* $D(P) \in \Gamma$ as follows:

$$D(P) := \sum_{i=1}^{r} l_i \otimes \alpha_i.$$

EXAMPLE 2.4.3. Let P be a cube with side a. The cube has 12 edges, each has length a and angle $\frac{\pi}{2}$. So we have

$$D(P) = \sum_{i=1}^{12} a \otimes \frac{\pi}{2} = a \otimes \left(12\frac{\pi}{2}\right) = a \otimes (6\pi) = a \otimes 0 = 0.$$

We will prove two lemmas:

LEMMA 2.4.4. *If P is cut into polyhedral pieces $P_1, \dots, P_s$ then*

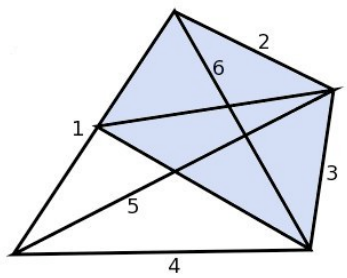
$$D(P) = D(P_1) + \ldots + D(P_s).$$

LEMMA 2.4.5. *If Q is a regular tetrahedron then $D(Q) \neq 0$.*

COROLLARY 2.4.6. *The Hilbert's third problem has a negative solution.*

Indeed, if P is a cube then Lemma 2.4.3 shows that $D(P) = 0$. If P is cut into polyhedral pieces $P_1, \dots, P_s$ then $D(P_1) + \ldots + D(P_s) = 0$ by Lemma 2.4.4. If Q is a regular tetrahedron then $D(Q) \neq 0$ by Lemma 2.4.5. So by Lemma 2.4.4, we can not rearrange pieces $P_1, \dots, P_s$ to get Q.

Proof of Lemma 2.4.4. A complete proof is a bit tedious, so we will just give a proof "by example" that completely explains what's going on.



Let P be a tetrahedron cut into two tetrahedra, a blue P' and a white P''. The polytope P has six edges of length $l_1, \dots, l_6$ and with angles $\alpha_1, \dots, \alpha_6$:

$$D(P) = \sum_{i=1}^{6} l_i \otimes \alpha_i.$$

The first edge of P is cut between P' and P'', let l_1' and l_1'' be the legths of the corresponding edges. Notice that $l_1 = l_1' + l_1''$. Likewise, the third angle α_3 is the sum of angles α_3' and α_3''. Also, P' and P'' share two new edges, of lengths m_1 and m_2 and with angles β_1', β_1'', β_2' and β_2''. Notice that $\beta_1' + \beta_1'' = \pi$ and $\beta_2' + \beta_2'' = \pi$. Now we are ready for bookkeeping:

$$D(P') = l_1' \otimes \alpha_1 + l_2 \otimes \alpha_2 + l_3 \otimes \alpha_3' + l_6 \otimes \alpha_6 + m_1 \otimes \beta_1' + m_2 \otimes \beta_2'$$

$$D(P'') = l_1'' \otimes \alpha_1 + l_4 \otimes \alpha_4 + l_3 \otimes \alpha_3'' + l_5 \otimes \alpha_5 + m_1 \otimes \beta_1'' + m_2 \otimes \beta_2''$$

Adding $D(P')$ and $D(P'')$ together, we get

$$(l_1' + l_1'') \otimes \alpha_1 + l_2 \otimes \alpha_2 + l_3 \otimes (\alpha_3' + \alpha_3'') + l_4 \otimes \alpha_4 + l_5 \otimes \alpha_5 + l_6 \otimes \alpha_6 +$$

$$m_1 \otimes (\beta_1' + \beta_1'') + m_2 \otimes (\beta_2' + \beta_2'') =$$

$$l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + l_3 \otimes \alpha_3 + l_4 \otimes \alpha_4 + l_5 \otimes \alpha_5 + l_6 \otimes \alpha_6 + m_1 \otimes \pi + m_2 \otimes \pi =$$

$$l_1 \otimes \alpha_1 + l_2 \otimes \alpha_2 + l_3 \otimes \alpha_3 + l_4 \otimes \alpha_4 + l_5 \otimes \alpha_5 + l_6 \otimes \alpha_6 = D(P).$$

We see that Lemma basically follows from the bilinearity of the tensor product and from the fact that each time cutting creates new edges, the sum of angles at these edges adds up to a multiple of π. □

Proof of Lemma 2.4.5. Let Q be a regular hexagon wigth side a. By the Law of Cosines, the angle between its faces is equal to $\arccos\frac{1}{3}$. So we have

$$D(Q)=\sum_{i=1}^{6} a\otimes\arccos\frac{1}{3}=(6a)\otimes\arccos\frac{1}{3}.$$

CLAIM 2.4.7. *$a\otimes\alpha=0$ in $\mathbb{R}\otimes(\mathbb{R}/\pi\mathbb{Z})$ if and only if either $a=0$ or $\alpha\in\mathbb{Q}\pi$.*

Proof. We certainly have $0\otimes\alpha=0$. If $\alpha=\frac{m}{n}\pi$ then

$$a\otimes\alpha=a\otimes\frac{m}{n}\pi=\left(n\frac{a}{n}\right)\otimes\left(\frac{m}{n}\pi\right)=\frac{a}{n}\otimes\left(n\frac{m}{n}\pi\right)=\frac{a}{n}\otimes(m\pi)=0.$$

Now we prove another implication. Fix $a_0\neq 0$ and $\alpha_0\neq\frac{m}{n}\pi$. Consider $\mathbb{R}$ as a $\mathbb{Q}$-vector space. Then a_0 spans a 1-dimensional subspace $L=\mathbb{Q}a_0$. We have a $\mathbb{Q}$-linear function $L\to\mathbb{Q}$ that sends a_0 to 1. This function can be extended to $\mathbb{Q}$-linear function $l:\ \mathbb{R}\to\mathbb{Q}$ that sends a_0 to 1.

We have a $\mathbb{Z}$-bilinear function

$$\mathbb{R}\times\mathbb{R}\to\mathbb{R},\quad (a,\alpha)\mapsto l(a)\alpha.$$

$\mathbb{R}$ contains $\pi\mathbb{Q}$ as a $\mathbb{Z}$-submodule. Composing a map above with the projection $\mathbb{R}\to\mathbb{R}/(\pi\mathbb{Q})$, we get a $\mathbb{Z}$-bilinear function

$$\mathbb{R}\times\mathbb{R}\to\mathbb{R}/(\pi\mathbb{Q}),\quad (a,\alpha)\mapsto l(a)\alpha+\pi\mathbb{Q}.$$

Notice that any pair of the form $(a,\pi n)$ is mapped to 0 (because $l(a)$ is a rational number), so our function induces a $\mathbb{Z}$-bilinear function

$$\mathbb{R}\times(\mathbb{R}/\pi\mathbb{Z})\xrightarrow{\beta}\mathbb{R}/(\pi\mathbb{Q}),\quad (a,\alpha)\mapsto l(a)\alpha+\pi\mathbb{Q}.$$

By the universal property of the tensor product, this bilinear map factors through the tensor product:

$$\begin{array}{ccc}\mathbb{R}\times(\mathbb{R}/\pi\mathbb{Z}) & \xrightarrow{\quad\beta\quad} & \mathbb{R}/(\pi\mathbb{Q})\\ & \searrow\qquad\nearrow & \\ & \mathbb{R}\otimes_{\mathbb{Z}}(\mathbb{R}/\pi\mathbb{Z}) & \end{array}$$

We have

$$\beta(a_0,\alpha_0)=l(a_0)\alpha_0+\pi\mathbb{Q}=\alpha_0+\pi\mathbb{Q}\neq 0.$$

Therefore,

$$a_0\otimes\alpha_0\neq 0.$$

This shows the first Claim. □

CLAIM 2.4.8. *If $\cos\frac{2\pi m}{n}\in\mathbb{Q}$ then it is equal to 1, $\frac{1}{2}$, 0, $-\frac{1}{2}$, or -1. In particular,*

$$\arccos\frac{1}{3}\notin\mathbb{Q}\pi.$$

Proof. Suppose $\cos\frac{2\pi m}{n}\in\mathbb{Q}$. We can assume that m and n are coprime. Let

$$\xi=\cos\frac{2\pi m}{n}+i\sin\frac{2\pi m}{n}\in\mathbb{C}.$$

Then ξ is a primitive n-th root of 1. Let $\mathbb{Q}(\xi)$ be the minimal field containing ξ (a cyclotomic field) and let $[\mathbb{Q}(\xi):\mathbb{Q}]$ be the degree of this field extension, i.e. the dimension of $\mathbb{Q}(\xi)$ over $\mathbb{Q}$. Then

$$\mathbb{Q}(\xi)\subset\mathbb{Q}\left(i\sin\frac{2\pi m}{n}\right)=\mathbb{Q}\left(\sqrt{\cos^2\frac{2\pi m}{n}-1}\right)=\mathbb{Q}(\sqrt{r}),$$

where r is a rational number. So $\mathbb{Q}(\xi)$ is at most a quadratic extension of $\mathbb{Q}$, and therefore,

$$[\mathbb{Q}(\xi):\mathbb{Q}]=1 \text{ or } 2.$$

On the other hand, a basic fact from the Galois theory that we are going to take on faith here is that

$$[\mathbb{Q}(\xi):\mathbb{Q}]=\phi(n),$$

where an Euler function $\phi(n)$ counts how many numbers between 0 and n are coprime to n, i.e. how many elements of the ring $\mathbb{Z}/n\mathbb{Z}$ are invertible. Take a prime decomposition

$$n=p_1^{k_1}\dots p_s^{k_s}.$$

By the Chinese theorem on remainders, we have an isomorphism of rings

$$\mathbb{Z}/n\mathbb{Z}=\mathbb{Z}/p_1^{k_1}\oplus\dots\oplus\mathbb{Z}/p_s^{k_s}.$$

This isomorphism induces an isomorphism of groups of invertible elements

$$(\mathbb{Z}/n\mathbb{Z})^*=(\mathbb{Z}/p_1^{k_1})^*\times\dots\times(\mathbb{Z}/p_s^{k_s})^*.$$

This gives a formula

$$\phi(n)=\phi(p_1^{k_1})\dots\phi(p_s^{k_s}).$$

It is clear that $\phi(p^k)=p^k-p^{k-1}$ because a number is coprime to p^k if and only if it is coprime to p, and $\mathbb{Z}/p^k\mathbb{Z}$ contains exactly p^{k-1} elements that are divisible by p. So we have

$$\phi(n)=(p_1^{k_1}-p_1^{k_1-1})\dots(p_s^{k_s}-p_s^{k_s-1})=$$

$$=p_1^{k_1-1}\dots p_s^{k_s-1}\times(p_1-1)\dots(p_s-1).$$

If $\phi(n)\leq 2$ then each $p_i\leq 3$ and each $k_i\leq 2$. Going through the list of possibilities, we see that the only solutions are

$$n=1,2,3,4,6.$$

This gives the Claim. □

Combining two claims finishes the proof of the Hilbert's 3d problem. □

§2.5. **Right-exactness of a tensor product.** Let's fix an R-module M and study the operation of "tensoring with M":

$$N \mapsto N \otimes_R M.$$

This gives a map from the category of R-modules to itself. Moreover, for any R-linear map $N \xrightarrow{f} N'$, we can define an R-linear map

$$N \otimes_R M \xrightarrow{f \otimes \mathrm{Id}} N' \otimes_R M, \quad n \otimes m \mapsto f(n) \otimes m.$$

Of course pure tensors are not linearly independent, so we have to check that $f \otimes \mathrm{Id}$ is well-defined. This can be done as follows. We have a map

$$N \times M \to N' \otimes_R M, \quad (n, m) \mapsto f(n) \otimes m,$$

which is clearly bilinear. So, by the universal property of the tensor product, it gives a linear map

$$N \otimes_R M \to N' \otimes_R M,$$

which is exactly our map $f \otimes \mathrm{Id}$.

LEMMA 2.5.1. *"Tensoring with M" functor $\cdot \otimes_R M$ is a functor from the category of R-modules to itself.*

Proof. To show that something is a functor, we have to explain how it acts on objects and morphisms in the category (this is done above), and then check axioms of a functor. There are two axioms: a functor should preserve identity maps and compositions of maps.

This is an example of a calculation that's much easier to do in your head than to read about. Still, let's give a proof just to show how it's done.

If $N \to N$ is an identity map, then $N \otimes_R M \xrightarrow{\mathrm{Id} \otimes \mathrm{Id}} N \otimes_R M$ is also obviously an identity map.

Suppose we have maps $N \xrightarrow{f} N' \xrightarrow{g} N''$. Let's compute the composition

$$N \otimes_R M \xrightarrow{f \otimes \mathrm{Id}} N' \otimes_R M \xrightarrow{g \otimes \mathrm{Id}} N'' \otimes_R M.$$

It takes a pure tensor $n \otimes m$ to the pure tensor $f(n) \otimes m$ and then to the tensor $g(f(n)) \otimes m = (g \circ f)(n) \otimes m$. The map

$$N \otimes_R M \xrightarrow{(g \circ f) \otimes \mathrm{Id}} N'' \otimes_R M$$

has the same effect on pure tensors. Since pure tensors span $N \otimes_R M$, we see that

$$(g \otimes \mathrm{Id}) \circ (f \otimes \mathrm{Id}) = (g \circ f) \otimes \mathrm{Id},$$

which exactly means that tensoring with M preserves composition. □

LEMMA 2.5.2. *There exists a canonical isomorphism $R \otimes_R M \simeq M$, $r \otimes m \mapsto rm$.*

Proof 1. For any R-module K, an R-bilinear map $R \times M \xrightarrow{F} K$ defines an R-linear map $M \xrightarrow{f} K$ by formula $f(m) = F(1, m)$. And vice versa, an R-linear map $M \xrightarrow{f} K$ defines an R-bilinear map $R \times M \xrightarrow{F} K$ by formula $F(r, m) = rf(m)$. This gives a natural (in K) bijection between bilinear maps $R \times M \to K$ and linear maps $M \to K$. It follows that functors *BilMaps*$_{R,M}$ and h_M are naturally isomorphic. It follows that functors

$h_{R\otimes_R M}$ and h_M are naturally isomorphic. By Yoneda's lemma, it follows that $R\otimes_R M$ and M themselves are isomorphic. To see that this isomorphism has the form $r\otimes m\mapsto rm$, recall that the proof of Yoneda's lemma is constructive: to find an isomorphism we have to apply the natural transformation to the identity morphism. So take $K=M$ and $f=\mathrm{Id}_M$ in the analysis above. Then $F(r,m)=rm$. □

Proof 2. Define a bilinear map $R\times M\to M$ by formula $(r,m)\mapsto rm$. By the universal property of the tensor product, it factors through a linear map

$$R\otimes_R M\xrightarrow{B} M,\quad r\otimes m\mapsto rm.$$

This map is clearly surjective (take $r=1$). Take a tensor $\sum_i r_i\otimes m_i\in\operatorname{Ker}B$. Then $\sum_i r_im_i=0$. It follows that

$$\sum_i r_i\otimes m_i=\sum_i r_i(1\otimes m_i)=\sum_i 1\otimes(r_im_i)=$$

$$=1\otimes\left(\sum_i r_im_i\right)=1\otimes 0=0.$$

So B is also injective. □

Now the main result:

THEOREM 2.5.3. *$\cdot\otimes_R M$ is a right-exact functor, i.e. for any exact sequence*

$$N'\xrightarrow{f}N\xrightarrow{g}N''\to 0, \tag{9}$$

the induced sequence

$$N'\otimes_R M\xrightarrow{f\otimes\mathrm{Id}}N\otimes_R M\xrightarrow{g\otimes\mathrm{Id}}N''\otimes_R M\to 0$$

is also exact.

Proof. To show that $g\otimes\mathrm{Id}$ is surjective, it suffices to show that any pure tensor $n''\otimes m\in N''\otimes_R M$ is in the image of $g\otimes\mathrm{Id}$. But g is surjective, so $n''=g(n)$ for some n, and then $n''\otimes m=g(n)\otimes m$.

Next we show that

$$\operatorname{Im}(f\otimes\mathrm{Id})\subset\operatorname{Ker}(g\otimes\mathrm{Id}).$$

Indeed, any tensor in the image of $f\otimes\mathrm{Id}$ can be written as $\sum_i f(n_i')\otimes m$. Applying $g\otimes\mathrm{Id}$, we get

$$\sum_i g(f(n_i'))\otimes m=\sum_i 0\otimes m=0.$$

The only non-trivial calculation is to show that

$$\operatorname{Ker}(g\otimes\mathrm{Id})\subset\operatorname{Im}(f\otimes\mathrm{Id}).$$

Consider a bilinear map

$$\beta:\ N\times M\to N\otimes_R M\to(N\otimes_R M)/\operatorname{Im}(f\otimes\mathrm{Id}),$$

where the second map is just a projection. For any $n'\in N'$, we

$$\beta(f(n'),m)=f(n')\otimes m+\operatorname{Im}(f\otimes\mathrm{Id})=0.$$

So β induces a bilinear map

$$\tilde{\beta}:\ (N/\operatorname{Im} f)\times M\to (N\otimes_R M)/\operatorname{Im}(f\otimes \mathrm{Id})$$

by a well-defined formula

$$\tilde{\beta}(n+\operatorname{Im} f, m):=\beta(n,m).$$

Since (9) is exact, we have

$$N/\operatorname{Im} f\simeq N/\operatorname{Ker} g\simeq N''.$$

So $\tilde{\beta}$ induces a bilinear map

$$\tilde{\beta}:\ N''\times M\to (N\otimes_R M)/\operatorname{Im}(f\otimes \mathrm{Id}),$$

which operates as follows: for any pair (n'',m), write $n''=g(n)$, then

$$\tilde{\beta}(n'',m)=n\otimes m+\operatorname{Im}(f\otimes \mathrm{Id}).$$

By the universal property of the tensor product, $\tilde{\beta}$ factors through the linear map

$$\tilde{B}:\ N''\otimes_R M\to (N\otimes_R M)/\operatorname{Im}(f\otimes \mathrm{Id})$$

such that

$$\tilde{B}(g(n)\otimes m)=n\otimes m+\operatorname{Im}(f\otimes \mathrm{Id}).$$

The main point is that $\tilde{B}$ is a well-defined map. Here is the main calculation: take $\sum_i n_i\otimes m\in\operatorname{Ker}(g\otimes \mathrm{Id})$, i.e. $\sum_i g(n_i)\otimes m=0$. Then

$$\tilde{B}\left(\sum_i g(n_i)\otimes m\right)=\tilde{B}(0)=0.$$

But on the other hand,

$$\tilde{B}(\sum_i g(n_i)\otimes m)=\sum_i \tilde{B}(g(n_i)\otimes m)=\sum_i n_i\otimes m+\operatorname{Im}(f\otimes \mathrm{Id}).$$

It follows that

$$\sum_i n_i\otimes m\in\operatorname{Im}(f\otimes \mathrm{Id}),$$

and so $\operatorname{Ker}(g\otimes \mathrm{Id})\subset\operatorname{Im}(f\otimes \mathrm{Id})$. □

Right-exactness is a very useful tool for computing tensor products.

PROPOSITION 2.5.4. *Suppose N is a finitely presented R-module, i.e. we have an exact sequence*

$$R^n\xrightarrow{A}R^m\to N\to 0,$$

where A is an $m\times n$ matrix of elements of R. Then

$$N\otimes M\simeq M^m/\operatorname{Im}[M^n\xrightarrow{A}M^m],$$

where an R-linear map $A:\ M^n\xrightarrow{A}M^m$ just multiplies a column vector of n elements of M by a matrix A.

Proof. This immediately follows from Lemma 2.5.2 and right-exactness of a tensor product. Indeed, exactness of the presentation of N implies exactness of the sequence

$$M^n \xrightarrow{A} M^m \to N \otimes M \to 0$$

and Proposition follows. □

EXAMPLE 2.5.5. Let's compute $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_9$. Take a presentation for $\mathbb{Z}_6$:

$$\mathbb{Z} \xrightarrow{\cdot 6} \mathbb{Z} \to \mathbb{Z}_6 \to 0$$

and tensor it with $\mathbb{Z}_9$:

$$\mathbb{Z}_9 \xrightarrow{\cdot 6} \mathbb{Z}_9 \to \mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_9 \to 0.$$

So $\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_9$ is isomorphic to the quotient of $\mathbb{Z}_9$ by a submodule of multiples of 6. Since $g.c.d.(6, 9) = 3$, this is the same thing as the quotient of $\mathbb{Z}_9$ by a submodule of multiples of 3. So

$$\mathbb{Z}_6 \otimes_{\mathbb{Z}} \mathbb{Z}_9 \simeq \mathbb{Z}_9/3\mathbb{Z}_9 \simeq \mathbb{Z}_3.$$

§2.6. **Restriction of scalars.** Recall that if V is a complex vector space, we can also consider V as a real vector space by "forgetting" how to multiply by $i \in \mathbb{C}$. This gives a forgetful functor

$$\mathbf{Vect}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{R}},$$

called restriction of scalars. Restriction of scalars doubles dimension: if $\{e_1, \dots, e_n\}$ is a basis of V (over $\mathbb{C}$) then the basis of V over $\mathbb{R}$ is given by

$$\{e_1, \dots, e_n, ie_1, \dots, ie_n\}$$

We can define restriction of scalars in a much broader setting of modules. Consider an arbitrary homomorphism of rings

$$f : R \to S$$

(in the example above, this is just an inclusion of fields $\mathbb{R} \hookrightarrow \mathbb{C}$). Suppose M is an S-module. We claim that we can also view M as an R-module, by keeping an old structure of an Abelian group on M, and defining an action of an element $r \in R$ on $m \in M$ by formula

$$(r, m) \mapsto f(r)m.$$

It is easy to see that this endows M with a structure of an R-module: an expression $f(r)m \in M$ is bilinear in both r and m, and also we have

$$f(r_1 r_2)m = [f(r_1)f(r_2)]m = f(r_1)\left(f(r_2)m\right).$$

Also, for any S-linear map of S-modules $M_1 \to M_2$, the same map is also automatically R-linear, and so we get a "restriction of scalars" functor

$$\mathbf{Mod}_S \to \mathbf{Mod}_R.$$

EXAMPLE 2.6.1. The map $\mathbb{R} \hookrightarrow \mathbb{C}$ is an inclusion, but restriction of scalars is also very interesting in the opposite case when $f : R \to S$ is surjective, i.e. when $S \simeq R/I$, where $I \subset R$ is some ideal. We can ask, which R-modules can be obtained by restricting of scalars from R/I-modules? In other words, which R-modules M can also be viewed as R/I-modules? The condition is simple: I should act trivially on M, i.e. we should have

$rm = 0$ for any $r \in I$, $m \in M$. For example, modules over $\mathbb{Z}/4\mathbb{Z}$ can be identified with $\mathbb{Z}$-modules (i.e. Abelian groups) where 4 acts trivially. For instance, if this module is finitely generated, then the structure theorem of finitely generated Abelian groups implies that M is a direct sum of several copies of $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$.

§2.7. **Extension of scalars.** Going back to complex and real vector spaces, we have a much more interesting functor

$$\mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{C}},$$

called *complexification*, or extension of scalars, which is defined as follows. For any vector space V over $\mathbb{R}$, consider the set of pairs of vectors (v_1, v_2), which we are going to write as "formal" linear combinations $v_1 + iv_2$, and define the multiplication by $r = a + bi \in \mathbb{C}$ as usual:

$$(a + bi)(v_1 + iv_2) = (av_1 - bv_2) + i(av_2 + bv_1).$$

It is easy to see that this gives a vector space $V_{\mathbb{C}}$ over $\mathbb{C}$ called complexification of V. For example, if V is a vector space of real column vectors then $V_{\mathbb{C}}$ is a vector space of complex column-vectors.

Moreover, for any $\mathbb{R}$-linear map $V \xrightarrow{f} V'$, we have an induced $\mathbb{C}$-linear map $V_{\mathbb{C}} \to V'_{\mathbb{C}}$ that sends $v_1 + iv_2$ to $f(v_1) + if(v_2)$. So the complexification is indeed a functor $\mathbf{Vect}_{\mathbb{R}} \to \mathbf{Vect}_{\mathbb{C}}$. Notice that if $\{e_1, \ldots, e_n\}$ is a basis of V (over $\mathbb{R}$) then $\{e_1, \ldots, e_n\}$ is also a basis of $V_{\mathbb{C}}$ (over $\mathbb{C}$), i.e. complexification preserves dimensions. However, the basis of $V_{\mathbb{C}}$ over $\mathbb{R}$ is equal to $\{e_1, \ldots, e_n, ie_1, \ldots, ie_n\}$, and so $V_{\mathbb{C}}$ over $\mathbb{R}$ has the same dimension as the tensor product $V \otimes_{\mathbb{R}} \mathbb{C}$, because $\mathbb{C}$ (as a vector space over $\mathbb{R}$) has basis $\{1, i\}$. In fact, $V_{\mathbb{C}}$ (as a real vector space) is isomorphic to $V \otimes_{\mathbb{R}} \mathbb{C}$. This isomorphism is independent of the choice of basis and simply takes $v_1 + iv_2$ to $v_1 \otimes 1 + v_2 \otimes i$. However, $V \otimes_{\mathbb{R}} \mathbb{C}$ is just a real vector space but $V_{\mathbb{C}}$ is a complex vector space. Is it possible to introduce the structure of a compex vector space on $V \otimes_{\mathbb{R}} \mathbb{C}$ directly?

We will see that this is easy, and can be done in a framework of modules. Consider an arbitrary homomorphism of rings

$$f : R \to S$$

(in the example above, this was an inclusion of fields $\mathbb{R} \hookrightarrow \mathbb{C}$). Suppose M is an R-module and we want to construct an S-module. First of all, notice that S, as any other S-module, can be viewed as an R-module by "restriction of scalars" construction above. So we can form a tensor product

$$M \otimes_R S$$

This is not yet what we want, because $M \otimes_R S$ is an R-module, but we want an S-module. So we are going to define the action of S on $M \otimes_R S$ by, as usual, defining it on pure tensors first by formula

$$(s, m \otimes s') \mapsto m \otimes (ss')$$

LEMMA 2.7.1. *This gives a well-defined S-module structure on $M \otimes_R S$, called the extension of scalars from M.*

Proof. Why is this well-defined? Consider an R-bilinear map

$$M \times S \to M \otimes_R S, \quad (m, s') \mapsto m \otimes (ss')$$

By linear extension, it gives an R-linear map

$$M \otimes_R S \to M \otimes_R S, \quad m \otimes s' \mapsto m \otimes (ss'),$$

which is exactly what we want.

The only thing to check is that this indeed gives an action of S, i.e. that all axioms of an S-module are satisfied. Our action on arbitrary tensors is

$$\left(s, \sum_i m_i \otimes s'_i\right) \mapsto \sum_i m_i \otimes (ss'_i).$$

This is bilinear both in s and in linear combinations $\sum_i m_i \otimes s'_i$. Finally, we have to check that the effect of multiplying by $s_1 s_2$ is the same as multiplying by s_2 and then multiplying by s_1. This is clear. □

§2.8. **Exercises.** In this worksheet, k is a field, R is a commutative ring, and p is a prime.

1. (a) Let $n, m \in \mathbb{Z}$ and let d be their g.c.d. Prove that

$$(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \simeq \mathbb{Z}/d\mathbb{Z}.$$

(b) Let R be a PID, let $x, y \in R$, and let d be their g.c.d. Prove that

$$(R/(x)) \otimes_R (R/(y)) \simeq R/(d).$$

2. An R-module M is called flat, if for any short exact sequence

$$0 \to N' \to N \to N'' \to 0$$

of R-modules, a sequence

$$0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is also exact. Classify all finitely generated flat $\mathbb{Z}$-modules.

3. Let V be a vector space over k. Show that V is a flat k-module.

4. Let M be an R-module and let $I \subset R$ be an ideal. Prove that

$$M \otimes_R (R/I) \simeq M/(IM).$$

5. Compute $(x, y) \otimes_{k[x,y]} \big(k[x, y]/(x, y)\big)$.

6. Let $R \to S$ be a homomorphism of rings and let M, N be two S-modules. By restriction of scalars, we can also view M and N as R-modules. Show that if $M \otimes_R N = 0$ then $M \otimes_S N = 0$. Is the converse true?

7. Let M and N be finitely generated modules over the ring of power series $k[[x]]$. Show that if $M \otimes_{k[[x]]} N = 0$ then either $M = 0$ or $N = 0$.

8. Consider linear maps of k-vector spaces $A : U \to V$ and $A' : U' \to V'$. We define their tensor product $A \otimes A'$ to be a linear map $U \otimes_k U' \to V \otimes V'$ such that

$$(A \otimes A')(u \otimes u') = A(u) \otimes A(u').$$

(a) Show that $A \otimes A'$ is well-defined. (b) Compute the Jordan normal form of $A \otimes A'$ if A and A' both have Jordan form $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. (c) Give a general formula for $\mathrm{Tr}(A \otimes A')$.

9. Let V be a finite-dimensional vector space and let V^* be its dual space. Construct a canonical isomorphism (independent of the basis) between (a) $V^* \otimes V^*$ and the vector space of bilinear maps $V \times V \to k$; (b) $V^* \otimes V$ and the vector space of linear maps $V \to V$.

10. Let M, N be two R-modules. Let $\mathrm{Hom}_R(M, N)$ be the set of R-linear maps $M \to N$. (a) Show that $\mathrm{Hom}_R(M, N)$ is an R-module. (b) Show that $\mathrm{Hom}(\cdot, M)$ is a left-exact contravariant functor from the category of R-modules to itself.

11. Let M_1, M_2, M_3 be R-modules. Construct a canonical isomorphism between $(M_1 \otimes_R M_2) \otimes_R M_3$ and $M_1 \otimes_R (M_2 \otimes_R M_3)$. Describe a covariant functor represented by this module without using a word "tensor".

12. Let R be a ring. An R-algebra S is a data that consists of a ring S and a homomorphism of rings $R \to S$. Then S is both a ring and an R-module (by restriction of scalars). For example, $k[x]$ is a k-algebra. (a) Show that if S_1, S_2 are two R-algebras then $S_1 \otimes_R S_2$ is also an R-algebra such that

$$(s_1 \otimes s_2)(s_1' \otimes s_2') = (s_1 s_1') \otimes (s_2 s_2')$$

(check that this multiplication is well-defined, satisfies all axioms of a commutative ring with 1, and there is a natural homomorphism $R \to S_1 \otimes_R S_2$.) (b) Prove that $k[x] \otimes_k k[y] \simeq k[x, y]$.

13. Construct a non-trivial Abelian group M such that $M \otimes_{\mathbb{Z}} M = 0$.