ALGEBRA 611: FINAL EXAM STUDY GUIDE

A final exam has been scheduled on W 12/16 at 10:30 in LGRT 1322. It will last 2 hours 30 minutes. No lecture notes or textbooks will be allowed during the final. Difficulty of the problems will match the midterms.

The test will contain 8 problems (problems will have multiple parts). You will have to select 6 problems that will be graded. There will be one problem for each topic outlined below (although some problems may refer to several topics at once). For each topic, I will list most important concepts, theorems, results, and techniques, which I hope you completely understand at this point. I will also give references for recommended reading.

Linear Algebra - I. The row-echelon form and the row-reduction algorithm: using it to solve practical problems and as the proof engine. Homogeneous system of linear equations with more variables than equations has a non-trivial solution. Determinants and their properties. Homogeneous system of linear equations with as many variables as equations has a non-trivial solution if and only if the determinant of its matrix is equal to zero. Vector spaces and linear maps in coordinates and without coordinates. Effect of the basis change on coordinates of vectors and on matrices of linear maps. The kernel and the image of a linear map. Row-rank is equal to the column-rank. Rank of the product does not exceed ranks of the terms. Linear independence, basis, dimension. Zorn's Lemma. Infinite-dimensional vector spaces: existence of a basis. Linear map can be extended from a linear subspace to the whole space. Dual vector space and the contragredient linear map. Dual basis. A vector space is canonically isomorphic to its second dual. Quotients and direct sums of vector spaces. Inverse matrix and adjoint matrix. Cramer's rule. If a system of linear equations with coefficients in the field K has a solution in the field extension of K then it also has a solution in K. Reading: Knapp I,II

Categories and Functors. Examples of categories. Examples of covariant and contravariant functors. Categories with one object. Posets. Examples of natural transformations. Examples of products and coproducts. Examples of categories without products or coproducts. Representable functors. Inverse limit of Abelian groups. Reading: Knapp IV.11, VI.6

Inner Product Spaces and Bilinear Forms. Inner product spaces: real symmetric and complex Hermitian. Matrix of the inner product. Change of the matrix under the basis change. Schwartz inequality. Orthonormal basis. Gram-Schmidt orthogonalization. Orthogonal and unitary groups. Adjoint and self-adjoint linear maps. Spectral Theorem. Polar decomposition. Bilinear form, its matrix. Change of matrix under the basis change. Radical of a bilinear form. Orthogonal complement. Principal Axis Theorem for symmetric forms. Signature of a real symmetric form. Sylvester theorem. Canonical form of an alternating bilinear form. If V has a nondegenerate alternating form then dim V is even. Reading: Knapp III, VI.1,2,3.

Groups and Group Actions. Vocabulary of group theory: group, homomorphism, subgroup, normal subgroup, coset, quotient group, center, centralizer, conjugacy class, normalizer of a subgroup. Lagrange theorem. Group actions. Transitive action. Orbit. Stabilizer. Counting with groups. Kernel and image of the homomorphism. First fundamental theorem. Second fundamental theorem. Cyclic groups. Orders of elements in a cyclic group. External and internal direct product of groups. Fundamental theorem on finite Abelian groups. Semi-direct product of groups. Automorphism of groups. Inner automorphisms. Class equation. p-groups. Group acts by conjugation on its subgroups. Sylow theorems. Groups of small order. Dihedral group. Symmetric group. Sign of a permutation. Cycle structure of a permutation. Alternating group. Group of invertible matrices (general linear group) over a finite field. Free groups. Subgroups of free groups. Reading: Knapp IV.1,2,3,6,7,9,10, VI.1,2.

Commutative Algebra - I. Vocabulary of commutative rings: rings, homomorphisms, subrings, ideals, quotient rings, principal ideals. Vocabulary of modules: R-modules, homomorphisms (R-linear maps), submodules, quotient modules. First and second isomorphism theorem for R-modules. Direct products and direct sums of R-modules. Integral domains and fields of fractions. Principal ideal domains. Unique factorization domains. Any PID is a UFD. Chinese remainder theorem. Characteristic of a field. Finite fields. Reading: Knapp IV.4,5, VIII.1,2,4

Linear Algebra - II. Linear algebra of free R-modules. Exact sequences of R-modules. Finitely presented R-modules. Smith normal form of a matrix over PID. Young diagrams. Structure theorem for finitely generated modules over PID. Corollary: structure theorem for finitely generated Abelian groups. Corollary: theory of linear operators acting on a finite-dimensional vector space. Linear operators in and without coordinates. Characteristic polynomial. Eigenvalues, eigenvectors, eigenspaces. Equivalence of the category of k[x]-modules and the category of k-vector spaces with a linear operator. Primary decomposition theorem for a linear operator. Cyclic normal form. Existence and uniqueness of the Jordan canonical form. Practical algorithm for computing the Jordan form. The Jordan form as a powerful proof engine. Diagonalization. Nilpotent operators. Minimal polynomial. Cayley-Hamilton theorem. Reading: Knapp VIII.6, V

Tensor Products. Tensor product of vector spaces. Tensor product of R-modules. Pure tensors. Showing that a tensor is non-trivial by using the universal property. Using the universal property of the tensor product in proofs. Canonical isomorphisms $M \otimes_R R \simeq M$, $M \otimes_R N \simeq N \otimes_R M$, etc. Computing the tensor product using finite presentations and right-exactness. Tensor product of finitely generated modules over PID. Restriction of scalars. Extension of scalars. Complexification. Reading: handout `http://www.math.umass.edu/~tevelev/alg2009_tensor.pdf`

Commutative Algebra - II. Ascending chain condition for rings and for modules. Noetherian rings. Any finitely generated module over a Noetherian ring admits a finite presentation. Hilbert Basis Theorem. Gauss Lemma. If R is a UFD then R[x] is a UFD. Prime ideals. Maximal ideals. Existence of maximal ideals. Irreducible polynomials: Eisenstein criterion. Characterizations of integral elements. Integral closure of a ring. Reading: Knapp VIII.3,5,8,9