ALGEBRA 612. TAKE-HOME MIDTERM.

The take-home midterm is due before class on Wednesday Apr 21. You can use lecture notes and a textbook (Knapp, or Atiyah–Macdonald, or any other textbook of your choosing). You should work independently: no outside help is allowed including internet.

I will ask each of you to present some of the problems (of my choice) during the review section on Monday April 26, so please be prepared.

Please select and solve 10 problems, and indicate clearly which problems you have picked. Only these problems will be graded. A problem with multiple parts (a), (b), etc. counts as one problem.

In this worksheet R denotes a ring (as usual, commutative and with 1) and k denotes a field.

1. Let R be a Noetherian ring. (a) Let $N \subset R$ be the nil-radical. Show that $N^n = 0$ for some n. (b) Let $f = a_0 + a_1x + a_2x^2 + \ldots \in R[[x]]$. Show that f is a nilpotent if and only if each a_i is a nilpotent of R.
2. (a) Suppose that $x^5 = x$ for any $x \in R$. Show that any prime ideal of R is maximal. (b) Let R be a local ring. Let $x \in R$. Show that if $x^2 = x$ then $x = 0$ or $x = 1$.
3. (a) Show that a sum of a unit and a nilpotent in R is a unit. (b) Let $f = a_0x^n + a_1x^{n-1} + \ldots + a_n \in R[x]$. Show that f is a unit in $R[x]$ if and only if a_n is a unit in R and each a_i for $i < n$ is a nilpotent in R.
4. (a) Let S be the set of all ideals in R in which every element is a zero-divisor. Show that this set contains maximal elements and each of them is a prime ideal. (b) The set of zero-divisors of R is a union of prime ideals.
5. (a) Let $A \to B$ be a homomorphism of rings such that B is integral over A. Show that the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ maps Zariski closed sets to Zariski closed sets. (b) Show that the integrality assumption in (a) is necessary.
6. Let A be an integral domain, let K be its field of fractions, and let R be the integral closure of A in K. Let f and g be monic polynomials in $K[x]$. (a) If $fg \in R[x]$ then $f, g \in R[x]$. (b) Show that it could happen that $fg \in A[x]$ but f or g is not in $A[x]$.
7. Let $(R, \mathfrak{m})$ be a local PID with the field of fractions K such that $R \neq K$. (a) Show that $k[[x]]$ is an example of such a ring. (b) Let $a \in K$. Show that either $a \in R$ or $\frac{1}{a} \in R$. (c) Show that there exists a unique homomorphism $v : K^* \to \mathbb{Z}$ with the following property: if $a \in R$ then $a \in \mathfrak{m}^{v(a)}$ but $a \notin \mathfrak{m}^{v(a)+1}$ (here we assume that $\mathfrak{m}^0 = R$).
8. Let $I, J \subset R$ be ideals. Prove that $I = J$ if and only if $I_\mathfrak{p} = J_\mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Spec} R$.
9. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal. Prove that I is radical if and only if one can write $I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in \operatorname{Spec} R$.
10. Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be closed algebraic sets. (a) Prove that $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m \simeq \mathbb{A}^{n+m}$ is also a closed algebraic set. (b) Let $f : X \to Y$

be a morphism of algebraic sets. Show that its graph

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$$

is a closed subset of $X \times Y$.

11. Let $(R, \mathfrak{m})$ be a Noetherian local ring. (a) Show that $x_1, \ldots, x_n \in R^n$ form a basis of R^n if and only if $x_1 + \mathfrak{m}^n, \ldots, x_n + \mathfrak{m}^n$ form a basis of an $(R/\mathfrak{m})$-vector space $R^n/\mathfrak{m}^n \simeq (R/\mathfrak{m})^n$. (b) Let A and B be finitely generated R-modules. Show that $A \oplus B$ is a free R-module if and only if A and B are free R-modules.

12. Describe explicitly irreducible components of an algebraic set

$$V(x^2 + y^2 + z^2, xy + xz + yz) \subset \mathbb{C}^3.$$

13. Let R be a Noetherian ring. Show that the set of minimal prime ideals of R is finite and their intersection is equal to the nilradical of R.