ALGEBRA 611: FINAL EXAM

No lecture notes or textbooks are allowed during the final. The test contains 8 problems (some problems have multiple parts). You have to select 6 problems that will be graded.

Please write your name here: ________________________________

Please select your six problems here: ________________________________

1. (a) Let A be a complex 4×7 matrix. Prove that $\mathrm{rk}(A) = 4$ if and only if there exists a complex 7×4 matrix B such that AB is the identity matrix. (b) Let A be an integer 4×7 matrix. Prove that A has rank 4 if and only if its reduction modulo p has rank 4 for infinitely many prime numbers p.

2. Let $F : \mathbf{Ab} \to \mathbf{Sets}$ be a *covariant* functor that sends an Abelian group G to the set of all pairs $F(G) = \{(x, y) \mid x, y \in G\}$ and that sends any homomorphism $f : G_1 \to G_2$ to the function that sends each pair $(x, y) \in F(G_1)$ to the pair $(f(x), f(y)) \in F(G_2)$. Show that F is a representable functor.

3. Let V be an $\mathbb{R}$-vector space of polynomials $f(x)$ of degree at most 1. Consider the following inner product on V:

$$(f, g) = \int_0^1 f(x)g(x)\,dx.$$

Let $T : V \to V$ be a differentiation linear map $T(f) = f'$ for any $f \in V$. Compute the adjoint linear operator of T.

4. Let P be a 5-Sylow subgroup of the symmetric group $G = S_{100}$. (a) Compute the order of P. (b) Let $H = N_G(P)$. Prove that $N_G(H) = H$.

5. Let R be a ring such that $x^2 = x$ for any $x \in R$. Show that if $P \subset R$ is a prime ideal then $R/P \simeq \mathbb{Z}/2\mathbb{Z}$.

6. (a) Find the characteristic and the minimal polynomials of

$$\begin{bmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Justify your answer. (b) Construct an explicit linear map $A : \mathbb{Q}^4 \to \mathbb{Q}^4$ with the following property. If $L \subset \mathbb{Q}^4$ is a $\mathbb{Q}$-vector subspace such that $A(L) \subset L$ then $L = \{0\}$ or $L = \mathbb{Q}^4$.

7. Fix ideals $I, J \subset R$. Prove that $(R/I) \otimes_R (R/J) \simeq R/(I + J)$.

8. Let R be a UFD with the field of fractions K. Let f, g, h be monic polynomials in $K[x]$ such that $fg = h$. Prove that $h \in R[x]$ if and only if $f, g \in R[x]$.