ALGEBRA 611, FALL 2009. TAKE-HOME MIDTERM.

The take-home midterm is due before class on Monday Nov 23. You can use lecture notes and the textbook. However, please do not discuss these problems with each other or use internet or other textbooks. I will ask each of you to present some of the problems (of my choice) during the review section on Monday at 4pm, so please be prepared. There is a "bail-out" provision: you can ask me not to grade *two* of the problems. A problem with multiple parts (a), (b), etc. counts as one problem. Please indicate clearly in the beginning of your exam which problems you don't wish to be graded. All other problems will be graded.

In this worksheet k denotes an arbitrary field, R denotes a ring (as usual, commutative and with 1), and p is a prime number.

1. Describe all ideals in the ring (a) $\mathbb{Z}[i]/(4)$; (b) $k[x,y]/(x^2,xy,y^2)$.

2. Let M be a $\mathbb{C}[x]$-module that corresponds in a usual way to a linear map $L:\mathbb{C}^4\to\mathbb{C}^4$ with a matrix

$$\begin{bmatrix}1&1&0&0\\0&1&0&0\\0&0&2&1\\0&0&0&2\end{bmatrix}.$$

Prove that M has a finite presentation and find it.

3. Let A be a square 3×3 matrix. Prove that if

$$\operatorname{Tr} A = \operatorname{Tr} A^2 = \operatorname{Tr} A^3 = 0$$

then A is nilpotent.

4. Let $R = k[[x]]$ be the ring of formal power series. Prove that the following categories C_1 and C_2 are equivalent. C_1 is the category of R-modules that are finite-dimensional as k-vector spaces. C_2 is the category of pairs (V,L), where V is a finite-dimensional k-vector space and L is a nilpotent operator on V (you have to define morphisms in C_2 yourself).

5. Let R be the ring of continuous real-valued functions on the closed interval $[0,1]$. (a) Prove that $f\in R$ is a zero-divisor if and only if f vanishes on some segment $[a,b]\subset[0,1]$, where $a<b$. (b) For any $x\in[0,1]$, let

$$m_x = \{f\in R \mid f(x)=0\}.$$

Prove that m_x is a maximal ideal of R. (c) Prove that any maximal ideal of R is given by m_x for some $x\in[0,1]$.

6. Let R be a PID and let M be a finitely generated R-module. Suppose that for any $m\in M$, there exists $r\in R\setminus\{0\}$ such that $rm=0$. For any irreducible element $p\in R$, and for any integer $i\geq 0$, let

$$M_{p,i} = \{m\in M \mid p^i m = 0\}.$$

(a) Prove that $M_{p,i}$ is an R-submodule of M and that $M_{p,i}/M_{p,i-1}$ is naturally an $R/(p)$-module. (b) For $i > 0$, let

$$n_{p,i} := \dim_{R/(p)} M_{p,i}/M_{p,i-1}.$$

Prove that these numbers uniquely determine an isomorphism class of M. (c) Let $R = \mathbb{C}[x]$ and let M be an R-module that corresponds (in a usual way) to a finite-dimensional $\mathbb{C}$-vector space V with a linear operator L. Let $p = x - \lambda$. Show that

$$n_{p,i} := \dim_{\mathbb{C}} \operatorname{Ker}(L - \lambda E)^i - \dim_{\mathbb{C}} \operatorname{Ker}(L - \lambda E)^{i-1}.$$

7. Suppose that the ring R has a property that $x^2 = x$ for any $x \in R$. (a) Show that $2x = 0$ for any $x \in R$. (b) Show that any ideal of R that can be generated by two elements in fact can be generated by one element. (c) Show that any finitely generated ideal of R is principal. (d) Give an example of a ring R such that $x^2 = x$ for any $x \in R$ but R contains non-principal ideals.

8. Classify all finitely generated modules (a) over the ring $\mathbb{Z}/4\mathbb{Z}$; (b) over the ring $k[x]/(x^3)$ (up to isomorphism).

9. Let $I, J \subset R$ be ideals. Prove that R/I is isomorphic to R/J (as R-modules) if and only if $I = J$. Is it true that R/I is isomorphic to R/J (as rings) if and only if $I = J$?

10. Let R^3 be a free R-module and suppose that $m_1, m_2, m_3 \in R^3$ generate R^3. Prove that m_1, m_2, m_3 is a basis of R^3.

11. Let G be a finite group and let $K \subset G$ be a normal subgroup. Let $P \subset K$ be a 149-Sylow subgroup of K. Show that $G = KN_G(P)$, where $N_G(P)$ is the normalizer of P in G.

12. Construct an explicit isomorphism between a 2-Sylow subgroup in $\mathrm{GL}_3(\mathbb{F}_2)$ and the dihedral group D_8.

13. Prove that any group of order 150 has a proper normal subgroup.

14. For any two real 3×3 matrices $A, B \in \mathrm{Mat}_{3,3}(\mathbb{R})$, let $(A, B) = \mathrm{Tr}(AB)$ be the trace of their product. (a) Show that this gives a symmetric bilinear form on $\mathrm{Mat}_{3,3}(\mathbb{R})$. (b) Compute its signature.