ALGEBRA 611, FALL 2009. HOMEWORK 7 [(1)]

In this worksheet k denotes an arbitrary field, R denotes a ring (as usual, commutative and with 1), and p is a prime number.

1. (a) Prove that the category of $\mathbb{Z}$-modules is equivalent to the category of Abelian groups. (b) Prove that the category of $k[x]$-modules is equivalent to the following category C: objects of C are pairs (V, L), where V is a k-vector space and $L : V \to V$ is a linear map. Morphisms of C are ... (you have to say what they are yourself).

2. Let C be the category of $\mathbb{Z}[i]$-modules and let $F : C \to \mathbf{Ab}$ be a forgetful functor. Find all prime numbers p such that $\mathbb{Z}/p\mathbb{Z}$ is in the image of F.

3. Let V be a finite-dimensional k-vector space and let $L : V \to V$ be a linear operator. Prove that there exists $i > 0$ such that $\operatorname{Ker}(L^i) \cap \operatorname{Im}(L^i) = 0$.

4. Let R be a PID, let M be a f.g. R-module, and let $r \in R$. Let $A_k = \{m \in M \mid r^k m = 0\}$ and let $B_k = \{m \in M \mid m = r^k m'\}$ for some $m' \in M$. Prove that $A_k \cap B_k = 0$ for some $k > 0$.

5. Let A be a non-singular square matrix. Show that there exists a polynomial $f \in k[x]$ such that $A^{-1} = f(A)$.

6. Let A be a complex $n \times n$ matrix such that every entry of A is equal to 1. Determine its (a) characteristic polynomial; (b) minimal polynomial; (c) Jordan normal form.

7. Let A be a complex square matrix. Show that there exist complex matrices D and N such that

- $A = D + N, \quad DN = ND$;
- D is diagonalizable and N is nilpotent, i.e. $N^s = 0$ for some s.

8. Let $\mathbb{F}_{p^n}$ be a finite field with p^n elements and let $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be a Frobenius automorphism $F(x) = x^p$. (a) Considering $\mathbb{F}_{p^n}$ as a vector space over $\mathbb{F}_p$, prove that F is $\mathbb{F}_p$-linear. (b) Find a minimal polynomial of F.

9. Ideals $I, J \subset R$ are called coprime if $I + J = R$. Prove that if I and J are coprime then I^7 and J^9 are also coprime.

10. Let $I = (x, y)$ be an ideal in $R = k[x, y]$. Considering I as an R-module, prove that it is finitely presented, and find its finite presentation.

11. Let R be a PID and let M be a finitely generated R-module. Give a definition of an element $r \in R$ such that if $R = k[X]$ and M corresponds to a linear operator $L : V \to V$, r is equal to the minimal polynomial of L.

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[1]This homework is due before class on Monday Nov 16. These problems will be discussed during the review section on Monday at 4pm. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. There is a "bail-out" provision: you can ask the grader not to grade *two* of the problems. Please indicate clearly in the beginning of your homework which problems you don't wish to be graded.