ALGEBRA 611, FALL 2009. HOMEWORK 6

1.
2. Compute the Smith normal form of the matrix
$$\begin{bmatrix} 4 & -4 & 4 \\ 4 & -3+i & 5+i \end{bmatrix}$$
over the Gaussian integers $\mathbb{Z}[i]$.
3. An R-module M is called torsion-free if $rm = 0$ (where $r \in R$ and $m \in M$) implies that $r = 0$ or $m = 0$. Prove that if R is a PID and M is a torsion-free finitely-generated module, then M is a free R-module.
4. (a) Prove that $\mathbb{Q}$ is a torsion-free but not free $\mathbb{Z}$-module.
   (b) Find a finitely-generated $k[x, y]$-module that is torision-free but not free.
5. Let $R = k[[x]]$ be the ring of formal power series. Show that R is a PID.
6. Let $R = k[[x]]$ be the ring of formal power series. Sow that isomorphism classes of finitely-generated R-modules are naturally parametrized by pairs (λ, r), where λ is a Young diagram and r is a non-negative integer.
7. (a) Show that an ideal (d) of a PID is maximal if and only if d is irreducible.
   (b) Describe all maximal ideals in $\mathbb{C}[x]$.
8. Describe all maximal ideals in $\mathbb{R}[x]$.
9.
10. Let $p \in \mathbb{Z}$ be a prime number such that $p \equiv 3 \mod 4$. Prove that $\mathbb{Z}[i]/(i)$ is a field with p^2 elements.
11. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules.
    (a) Prove that if M_1 and M_3 are torsion-free then M_2 is torsion-free.
    (b) Is the converse of (a) true?
12. Find four different maximal ideals in $\mathbb{Z}[\sqrt{-2}]$.