ALGEBRA 611, FALL 2009. HOMEWORK 4

This homework is due before the class on Monday October 26. These problems will be discussed during the review section on Monday at 4pm.

The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. Please make sure that all solutions are complete and accurately written.

There is a "bail-out" provision: you can ask the grader not to grade *two* of the problems. Please indicate clearly in the beginning of your homework which problems you don't wish to be graded.

In this worksheet, p and q always denote different prime numbers.

1. Describe all finite groups with exactly (a) two, (b) three conjugacy classes.

2. Let G be a p-group. Prove that there exists a sequence of subgroups

$$\{e\} = G_0 \subset G_1 \subset \ldots \subset G_n = G$$

such that G_i is normal in G_{i+1} and $G_{i+1}/G_i \simeq \mathbb{Z}/p\mathbb{Z}$ for each i.

3. (a) Find a Sylow p-subgroup G in $\mathrm{GL}_n(\mathbb{F}_p)$ (the group of invertible $n \times n$ matrices with coefficients in the finite field with p elements). (b) Compute its normalizer. (c) Find the sequence of subgroups of G as in Problem 2.

4. Find 2-Sylow subgroups in S_4 (the group of permutations of 4 letters) and 5-Sylow subgroups in A_5 (the group of even permutations of 5 letters). In particular, find the number of these Sylow subgroups.

5. Prove that one of Sylow subgroups in a group of order (a) 40, (b) p^2q is normal.

Definition. Let H, K be subgroups of G and suppose that

- H is normal in G;
- $HK = G$;
- $H \cap K = \{e\}$.

Then we say that G is an (inner) semidirect product of H and K.

6. (a) Prove that any group of order pq is a semidirect product of cyclic groups. (b) Prove that a dihedral group D_n (the group of all symmetries of the regular n-gon) is a semidirect product of cyclic groups $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

Definition. Let H and K be groups and let $\phi: K \to \mathrm{Aut}(H)$ be a homomorphism. An (outer) semidirect product of H and K is a set of pairs

$$\{(h,k) \,|\, h \in H,\ k \in K\}$$

with the following operation: $(h_1,k_1)(h_2,k_2) = (h_1\left[\phi(k_1)h_2\right], k_1k_2)$.

7. (a) Prove that an (outer) semidirect product is a well-defined group. (b) Prove that an (inner) semidirect product of H and K is isomorphic to their (outer) semidirect product with respect to some homomorphism ϕ.

8. Let P be a p-group and let $H \subset P$ be a normal subgroup of order p. Prove that H is contained in the center of P.

9. Prove that there exists a non-Abelian group of order p^3 and that this group can not be presented as a semidirect product of its proper subgroups.

10. (a) Show that if $H_1, H_2 \subset G$ are subgroups of finite index then $H_1 \cap H_2$ is also a subgroup of finite index. (b) Let $H \subset G$ be a subgroup of finite index. Show that H contains a subgroup N which is a normal subgroup of G of finite index.

11. Let I be a poset and suppose that I is directed, i.e. that for any two elements $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let G_i be an inverse system of groups indexed by I. Prove that the inverse limit exists.

12. Let I be the poset of normal subgroups of G of finite index (ordered by inclusion). Prove that quotient groups G/H form an inverse system of groups indexed by I and that this inverse system has an inverse limit (called the profinite completion of G).

13. Let G be a finite group acting on a finite set S. (a) Prove that the number of orbits is equal to

$$\frac{1}{|G|}\sum_{g\in G}|\operatorname{Fix}(g)|, \quad \text{where} \quad \operatorname{Fix}(g)=\{x\in S\,|\,gx=x\}.$$

(b) If $|S| > 1$ and the action is transitive (i.e. there is only one orbit), there exists $g \in G$ such that $\operatorname{Fix}(g)$ is empty. (c) Let H be a proper subgroup of a finite group G. Show that G is not the union of conjugates of H.

14. Let A be an Abelian group and let $\mathbf{Ab}/A$ be the category of Abelian groups over A, i.e. the category of homomorphisms $X \to A$ from arbitrary Abelian groups to A. Prove that this category has products and coproducts.