ALGEBRA 611, FALL 2009. HOMEWORK 2

This homework is due before the class on Monday September 28. These problems will be discussed during the review section on Monday at 4pm.

The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. Please make sure that all solutions are complete and accurately written.

There is a "bail-out" provision: you can ask the grader not to grade *two* of the problems. Please indicate clearly in the beginning of your homework which problems you don't wish to be graded.

1. Let P_3 be a vector space of complex polynomials in one variable of degree at most 3. (a) Prove that the formula

$$(f,g)=\int_{-1}^{1} f(x)\overline{g(x)}\,dx$$

gives an inner product on P_3. (b) Use Gram-Schmidt process to find an orthonormal basis of a subspace spanned by x and x^2. (c) Extend the basis of (b) to an orthonormal basis of P_3. (d) Let $D: P_3 \to P_3$ be a linear map given by differentiation, $f \mapsto f'$. Describe its adjoint linear map explicitly.

2. Let P be a vector space of all real polynomials in one variable. (a) Prove that the formula

$$(f,g)=\int_{-1}^{1} f(x)g(x)\,\frac{dx}{\sqrt{1-x^2}}$$

gives an inner product on P (be careful with indefinite integrals). (b) Prove that there exists a unique polynomial f_n of degree n such that

$$f_n(\cos\theta)=\cos(n\theta)$$

(Hint: induction). (c) Prove that $f_0, f_1, f_2, \ldots$ is an orthogonal basis of P.

3. Let U be a subspace of a vector space V with an inner product. Let

$$U^{\perp}=\{v\in V \mid (v,u)=0 \text{ for all } u\in U\}.$$

be an orthogonal complement. Prove that (a) if V is finite-dimensional then $V=U\oplus U^{\perp}$. (b) If V is infinite-dimensional then $U^{\perp}$ can be equal to 0 even if $U\neq V$!

4. (a) Let V be a finite-dimensional inner product space. Prove that a linear map $L: V\to V$ is an orthogonal projection onto a subspace if and only if L is self-adjoint and $L^2=L$. (b) In notation of Problem 1, give a matrix of an orthogonal projection onto a subspace in Problem 1 part (b) in the basis $\{1,x,x^2,x^3\}$.

5. Prove that the product of two Hermitian matrices is Hermitian if and only if they commute.

6. Prove that any matrix in $\mathrm{Mat}_{n,n}(\mathbb{R})$ (resp. $\mathrm{Mat}_{n,n}(\mathbb{C})$) is a product of an orthogonal (resp. unitary) matrix and an upper-triangular matrix.
7. Let A be a unitary matrix. Prove that all roots of $\det(A - z\,\mathrm{Id}) = 0$ lie on a unit circle $|z| = 1$.
8. Let A be a $k \times n$ complex matrix with $\mathrm{rk}\, A = n$. Prove that A^*A is an invertible matrix.
9. Let V be a finite-dimensional vector space with an inner product and let $f \in V^*$. Prove that there exists a unique $v_0 \in V$ such that $f(v) = (v, v_0)$ for any $v \in V$.
10. Consider $\mathbb{R}^n$ with the standard inner product. Let $H \subset \mathbb{R}^n$ be a subspace of codimension 1. (a) Prove that there exists a unique non-trivial orthogonal map $\mathbb{R}^n \to \mathbb{R}^n$ that fixes all vectors in H. It is called a reflection with respect to H. (b) Prove that any orthogonal map $\mathbb{R}^2 \to \mathbb{R}^2$ is a product of at most two reflections.
11. Let A be a Hermitian positive-semidefinite matrix. Prove that $A = B^*B$, where B is an upper-triangular matrix.
12. Let $v_1, \ldots, v_k \in V$, where V is an inner-product space. Consider the *Gram* matrix

$$G(v_1, \ldots, v_k) = \begin{bmatrix} (v_1, v_1) & \ldots & (v_1, v_k) \\ \vdots & \ddots & \vdots \\ (v_k, v_1) & \ldots & (v_k, v_k) \end{bmatrix}$$

(a) Prove that

$$[c_1 \ldots c_k] G(v_1, \ldots, v_k) \begin{bmatrix} \bar{c}_1 \\ \vdots \\ \bar{c}_k \end{bmatrix} = ||c_1 v_1 + \ldots + c_k v_k||^2.$$

Deduce that $G(v_1, \ldots, v_k)$ is Hermitian and positive semi-definite. (b) Show that $\det G(v_1, \ldots, v_k) \geq 0$ with equality if and only $v_1, \ldots, v_k$ are linearly dependent.
13. Prove that any symmetric real matrix can be written as QDQ^t, where Q is an orthogonal matrix and D is a diagonal matrix.
14. (a) Let V be a finite-dimensional real vector space with an inner product. Show that Problem 9 defines a canonical isomorphism between V and V^*. Let $L : V \to V$ be a linear operator. Show that an adjoint linear operator $L^* : V \to V$ can then be identified with a contragredient linear operator $L^* : V^* \to V^*$. (b) In the complex case, show that an identification between V and V^* given by Problem 9 is not an isomorphism.