ALGEBRA 612, SPRING 2010. HOMEWORK 8

In this worksheet the base field is always $\mathbb{C}$ unless otherwise stated.

1. Describe explicitly (i.e. not just dimensions but how each element of the group acts) all irreducible representations of (a) $(\mathbb{Z}/2\mathbb{Z})^r$; (b) D_{2n}.

2. Describe explicitly all irreducible representations and build a character table for (a) S_3; (b) A_4; (c) the quaternionic group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$.

3. (a) Let V be the standard $(n-1)$-dimensional representation of S_n and let sgn be the 1-dimensional sign representation. Find all n such that $V \simeq V \otimes \mathrm{sgn}$. (b) Describe explicitly all irreducible representations of S_4.

4. Let G be the group of affine transformations of $\mathbb{F}_7$ of the form $x \mapsto ax+b$, where $a, b \in \mathbb{F}_7$ and $a^3 = 1$. (a) Show that $|G| = 21$ and describe its conjugacy classes. (b) Describe explicitly all irreducible complex representations of G. (c) Let V be the 7-dimensional representation of G in the algebra of functions $\mathbb{F}_7 \to \mathbb{C}$ induced by the action of G on $\mathbb{F}_7$ by affine transformation. Decompose V as a direct sum of irreducible representations.

5. Let V be an irreducible representation of a finite group G over $\mathbb{R}$. (a) Show that the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$ is naturally a representation of G over $\mathbb{C}$. (b) This representation is either irreducible or a direct sum of two irreducible representations. (c) Show on examples that both possibilities in (b) do occur.

6. Let M be a module over a commutative ring R. Let $M^{\otimes k} = M \otimes_R \ldots \otimes_R M$ (k times). Let $\Lambda^k M$ be a quotient module of $M^{\otimes k}$ by a submodule generated by all elements of the form $x_1 \otimes \ldots \otimes x_k$ where $x_i = x_j$ for some i and j. For any decomposable tensor $x_1 \otimes \ldots \otimes x_k \in M^{\otimes k}$, its image in $\Lambda^k M$ is denoted by $x_1 \wedge \ldots \wedge x_k$. Show that if M is a free R-module of rank n with basis $e_1, \ldots, e_n$ then $\Lambda^k M$ is a free R-module of rank $\binom{n}{k}$ with basis $e_{i_1} \wedge \ldots \wedge e_{i_k}$ for all $1 \leq i_1 < \ldots < i_k \leq n$.

7. (a) Show that if V is a representation of G (over a field) then $\Lambda^k V$ is also a G-representation. (b) Let V_1 and V_2 be G-representations. Show that

$$\Lambda^k(V_1 \oplus V_2) \simeq \sum_{a+b=k} \Lambda^a V_1 \otimes \Lambda^b V_2.$$

8. (a) Let V be a representation of G with character χ_V. Show that

$$\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left(\chi_V(g)^2 - \chi_V(g^2)\right).$$

(b) Let V be the standard 4-dimensional irreducible representation of S_5. Show that $\Lambda^2 V$ is an irreducible 6-dimensional representation.

9. Let V be an irreducible representation of a finite group G. Show that there exists a unique G-invariant hermitian inner product on V.

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[0]This homework is due before class on Monday May 3. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. You can ask the grader not to grade *two* of the problems.

10. Show that the columns of the character matrix are orthogonal, and more precisely that

$$\sum_{\chi} \chi(g)\overline{\chi(g)} = \frac{|G|}{c(g)},$$

where the summation is over all irreducible characters of G and $c(g)$ is the number of elements of G conjugate to g. Also show that

$$\sum_{\chi} \chi(g)\overline{\chi(g')} = 0$$

if g and g' are not conjugate.

11. In this problem $k = \mathbb{F}_q$ is a finite field with $q = p^n$ elements. Let G be a p-group. Show that the trivial representation is the only irreducible representation of G over k.

12. For any two 2-dimensional representations V_1 and V_2 of D_{11}, decompose $V_1 \otimes V_2$ as a direct sum of irreducible representations.

13. Let V be a faithful representation of G (i.e. the homomorphism $G \to \mathrm{GL}(V)$ is injective). Show that any irreducible representation of G is contained in a tensor power $V^{\otimes n}$ for some n.