ALGEBRA 612, SPRING 2010. HOMEWORK 7

In this worksheet R is a ring and $k = \bar{k}$ is an algebraically closed field.

1. Let $f, g \in k[x_1, \ldots, x_n]$ be polynomials such that f is irreducible and $V(f) \subset V(g)$. Show that f divides g.

2. Let $\alpha : \mathbb{A}^2 \to \mathbb{A}^2$ be a morphism given by polynomials f and g in $k[x, y]$.
(a) Show that if α is an isomorphism then the polynomial

$$\det \begin{bmatrix} \frac{\partial f(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial x} \\ \frac{\partial f(x,y)}{\partial y} & \frac{\partial g(x,y)}{\partial y} \end{bmatrix}$$

is a nonzero constant (the converse of this is a famous open problem).
(b) Give an example when α is an isomorphism but polynomials f, g are not both linear polynomials.

3. Let $X \subset \mathbb{A}^2$ be defined by equations $x^2 + y^2 = 1$ and $x = 1$. Is it true that $I(X) = (f, g)$?

4. (a) Let $X \subset \mathbb{A}^2$ be a cuspidal cubic $x^2 = y^3$. Let $f : \mathbb{A}^1 \to X$ be defined by formulas $t \mapsto (t^3, t^2)$. Is it an isomorphism? (b) Is a hyperbola $xy = 1$ isomorphic to $\mathbb{A}^1$?

5. Consider the morphism $\mathbb{A}^2 \to \mathbb{A}^2$ defined by formulas $(x, y) \mapsto (x, xy)$. Is the image Zariski closed? Zariski open? Zariski dense?

6. Show that the category of irreducible algebraic sets and morphisms between them is equivalent to the category of finitely generated k-algebras without zero-divisors and homomorphisms between them.

7. Consider the morphism $\alpha : \mathbb{A}^1 \to \mathbb{A}^n$ given by $t \mapsto (t, t^2, \ldots, t^n)$. Show that α induces an isomorphism between $\mathbb{A}^1$ and $V(I)$, where I is generated by 2×2 minors of the following matrix

$$\begin{bmatrix} 1 & x_1 & x_2 & \ldots x_{n-1} \\ x_1 & x_2 & x_3 & \ldots x_n \end{bmatrix}$$

In the problems 8–11, let S be a multiplicative system in R.

8. Let M_1 and M_2 be submodules of an R-module N. Show that

$$(a) \qquad (S^{-1}M_1) + (S^{-1}M_2) = S^{-1}(M_1 + M_2);$$

$$(b) \qquad (S^{-1}M_1) \cap (S^{-1}M_2) = S^{-1}(M_1 \cap M_2);$$

$$(c) \qquad \text{if } M_1 \supset M_2 \text{ then } \quad S^{-1}(M_1/M_2) \simeq (S^{-1}M_1)/(S^{-1}M_2).$$

9. Let M be a finitely generated R-module. Let $\operatorname{Ann} M = \{r \in R \,|\, rM = 0\}$. Show that

$$S^{-1}(\operatorname{Ann} M) \simeq \operatorname{Ann}(S^{-1}M).$$

10. Let M_1 and M_2 be R-modules. Show that

$$S^{-1}(M_1 \otimes_R M_2) \simeq (S^{-1}M_1) \otimes_{S^{-1}R} (S^{-1}M_2).$$

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[0]This homework is due before class on Monday April 12. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. You can ask the grader not to grade *two* of the problems.

11. Show that the nilradical of $S^{-1}R$ is isomorphic to the localization of the nilradical of R.

12. (a) Let M be an R-module. Show that $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for any maximal ideal $\mathfrak{m} \subset R$. (b) The ring is called *reduced* if its nilradical is trivial. Show that the ring R is reduced if and only if $R_{\mathfrak{p}}$ s reduced for any prime ideal $\mathfrak{p} \subset R$.

13. Let R be an integral domain. For any R-module M, let

$$T(M) = \{x \in M \mid rx = 0 \text{ for some } r \in R\}$$

be the torsion submodule of M. (a) Show that $M \to T(M)$ is a left-exact functor $\mathrm{Mod}_R \to \mathrm{Mod}_R$. (b) Show that $T(M) = 0$ if and only if $T(M)_{\mathfrak{m}} = 0$ for any maximal ideal $\mathfrak{p} \subset R$.

14. Let R be a local Noetherian ring with a maximal ideal $\mathfrak{m}$. Show that

$$\bigcap_{n \geq 1} \mathfrak{m}^n = 0.$$

15. Show that the Nakayama's lemma fails if the module M is not assumed to be finitely generated.

16. Let M and N be finitely generated modules over a local ring R. Show that if $M \otimes_R N = 0$ then $M = 0$ or $N = 0$.

17. For any $f \in R$, let $D(f) \subset \operatorname{Spec} R$ be the complement of the closed set $V(f)$. (a) Show that sets $D(f)$ form a base of Zariski topology, i.e. any Zariski open subset of $\operatorname{Spec} R$ can be expressed as a union of open sets of the form $D(f)$. (b) Show that if $D(f) = D(g)$ then $R[1/f] \simeq R[1/g]$.

18. Let $x, y \in \operatorname{Spec} R$. (a) Show that there exists either a neighborhood of x that does not contain y or a neighborhood of y that does not contain x. (b) Show that the nilradical of R is a prime ideal if and only if $\operatorname{Spec} R$ contains a "generic point" η such that $\bar{\eta} = \operatorname{Spec} R$.

19. For any $\mathfrak{p} \in \operatorname{Spec} R$, let $k(\mathfrak{p})$ be the quotient field of $R/\mathfrak{p}$. This field is called the residue field at $\mathfrak{p}$. (a) Show that

$$k(\mathfrak{p}) \simeq R_p/\mathfrak{p}R_{\mathfrak{p}}.$$

(b) Let M be a finitely-generated R-module. The $k(\mathfrak{p})$-vector space

$$M \otimes_R k(\mathfrak{p})$$

is called *the fiber* of M at $\mathfrak{p}$. Show that if $M \otimes_R k(\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Spec} R$ then $M = 0$. (c) Let $R = \mathbb{Z}$ and let M be a finitely generated Abelian group. Compute all fibers of M on $\operatorname{Spec} Z$ and verify (b) in this case.