ALGEBRA 612, SPRING 2010. HOMEWORK 6

In this worksheet, k denotes a field and R denotes a commutative ring. Do not assume that k is algebraically closed unless otherwise stated.

1. Let $S \subset R$ be a multiplicative system. Consider the covariant functor $\mathrm{Rings} \to \mathrm{Sets}$ that sends a ring A to the set of all homomorphisms $f : R \to A$ such that $f(s)$ is a unit for any $s \in S$ (describe its action of homomorphisms $A \to A'$ yourself). Show that this functor is representable.

2. Show that a ring A is local if and only if its non-invertible elements form an ideal.

3. Let $f : A \to B$ be a homomorphism of rings. If I is an ideal of B then we define its contraction as $I^c = f^{-1}(I)$. If I is an ideal of A then we define its extension as $I^e = Bf(I)$. (a) Is it true that any ideal of B is extended? (b) Is it true that $(I^c)^e = I$ for any ideal I of B? (c) Show that the maps $I \to I^e$ and $I \to I^c$ induce a bijection between the set of contracted ideals in A and the set of extended ideals in B.

4. For any ideal I of R, let $V(I) = \{\mathfrak{p} \in \operatorname{Spec} R \,|\, I \subset \mathfrak{p}\}$. (a) Show that $\operatorname{Spec} R$ satisfies all axioms of a topological space with closed subsets $V(I)$. This topology is called *Zariski topology*. (b) Show that the pull-back $f^* : \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous in Zariski topology.

5. Let $S \subset R$ be a multiplicative subset and let $f : R \to S^{-1}R$ be a canonical homomorphism. (a) Show that the pull-back map $f^* \operatorname{Spec} S^{-1}R \to \operatorname{Spec} R$ is injective. (b) Let $I \subset R$ be an ideal. Show that $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$.

6. Let $\mathfrak{p} \subset R$ be a prime ideal. Show that the image of $\operatorname{Spec} R_{\mathfrak{p}}$ in $\operatorname{Spec} R$ is equal to the intersection of all open subsets in $\operatorname{Spec} R$ that contain a point $\mathfrak{p}$.

7. Let Σ be the set of all multiplicative subsets of R. Show that Σ contains maximal (by inclusion) subsets and that $S \in \Sigma$ is maximal if and only if $R \setminus S$ is a minimal prime ideal.

8. Describe (a) $\operatorname{Spec} \mathbb{Z}/n\mathbb{Z}$; (b) $\operatorname{Spec} \mathbb{Z}_5$ (5-adic numbers).

9. Let $I \subset R$ be an ideal. Show that

$$\sqrt{I} = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec} R \\ I \subset \mathfrak{p}}} \mathfrak{p}.$$

10. Let $\sqrt{0} \subset R$ be the nil-radical. Show that the canonical pull-back map $\operatorname{Spec} R/\sqrt{0} \to \operatorname{Spec} R$ is a homeomorphism (first show that it is a bijection of sets).

11. Suppose R is a direct product of rings $R_1 \times \ldots \times R_k$. Show that $\operatorname{Spec} R$ is homeomorphic to the disjoint union of spectra $\operatorname{Spec} R_1, \ldots, \operatorname{Spec} R_k$.

12. (a) Show that the intersection of closed subsets $\cap_\alpha V(I_\alpha)$ of $\operatorname{Spec} R$ is empty if and only if $\sum_\alpha I_\alpha = R$. (b) Show that $\operatorname{Spec} R$ is quasi-compact, i.e. any open covering of $\operatorname{Spec} R$ has a finite sub-covering.

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[0]This homework is due before class on Monday April 5. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. You can ask the grader not to grade *two* of the problems.

13. Let $f : A \to B$ be a homomorphism of rings. (a) Show that the pull-back $f^* : \operatorname{MaxSpec} B \to \operatorname{MaxSpec} A$ is not always well-defined. (b) If A and B are finitely generated k-algebras then the pull-back for $\operatorname{MaxSpec}$ is well-defined (do not assume that k is algebraically closed, although it helps to consider this case first).

14. Let $J, I_1, \ldots, I_r$ be ideals of R such that $J \subset I_1 \cup \ldots \cup I_r$. Show that J is in fact contained in one of the ideals I_k if any of the following conditions are satisfied. (a) $r = 2$. (b) All ideals $I_1, \ldots, I_r$ are prime (Hint: prove the contrapositive statement by induction on r).

15. Let $J, I_1, \ldots, I_r$ be ideals of R such that $J \subset I_1 \cup \ldots \cup I_r$. Show that J is in fact contained in one of the ideals I_k if any of the following conditions are satisfied. (a) At most two of the ideals $I_1, \ldots, I_r$ are not prime. (b) R contains an infinite field.