ALGEBRA 612, SPRING 2010. HOMEWORK 5

In this worksheet, k denotes a field. Do not assume that k is algebraically closed unless otherwise stated.

1. Let S be a finite set and let $\mathcal{I}$ be a non-empty collection of its subsets (called the independent sets). A pair $(S, \mathcal{I})$ is called a *finite matroid* if

- every subset of an independent set is independent;
- Exchange property: If A and B are independent sets and $|A| > |B|$ then there exists $x \in A \setminus B$ such that $B \cup \{x\}$ is independent.

Show that the following pairs are matroids. (a) S is the set of columns of a matrix A over a field k; $\mathcal{I}$ is the collection of linearly independent subsets of columns. (b) Let K/k be a field extension and let $S \subset K$ be a finite subset. Let $\mathcal{I}$ be the collection of algebraically independent subsets of S. (c) Let S be a set with n elements and let $\mathcal{I}$ be the collection of subsets with at most r elements (a uniform matroid). (d) Let S be the set of edges of a finite graph. Let $\mathcal{I}$ be the collection of subsets of edges that do not contain cycles. (Hint: if you are stuck with this problem, perhaps try to reread proofs of Lemmas 7.3.5. and 7.3.6. in the lecture notes).

2. Let $(S, \mathcal{I})$ be a finite matroid. A maximal (by inclusion) independent subset is called a basis of the matroid. Show that any two bases have the same number of elements (called the rank of the matroid).

3. (a) Define the notion of isomorphic matroids. (b) Let $(S, \mathcal{I})$ be a uniform matroid of rank 2 (as in 1c). Show that S is isomorphic to a matroid of type 1a with $k = \mathbb{F}_q$ if and only if $|S| \leq q + 1$. (c) Show that any matroid of type 1d is isomorphic to a matroid of type 1a with $k = \mathbb{F}_2$ (Hint: the matrix A has a simple meaning in terms of the graph: it has one column for each edge and one row for each vertex).

4. Let $\mathcal{B}$ be a non-empty collection of subsets of a finite set S. Show that $\mathcal{B}$ is a collection of bases of some matroid if and only if for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

5. Let x be transcendental over k and let $F \subset k(x)$ be a subfield properly contaning k. Show that $k(x)$ is finite-dimensional over F.

6. Let K_1 and K_2 be algebraically closed extensions of $\mathbb{C}$ of transcendence degree 11. Let $f : K_1 \to K_2$ be a homomorphism over $\mathbb{C}$. Show that f is an isomorphism.

7. Let $k \subset K \subset E$ be field extensions with $\text{tr.deg.}\, E/k < \infty$. Show that

$$\text{tr.deg.}\, E/k = \text{tr.deg.}\, K/k + \text{tr.deg.}\, E/K.$$

8. Prove the Noether normalization theorem when k is a finite field. Hint: instead of the substitution $y_i' = y_i - \lambda_i y_k$ (with $\lambda_i \in k$), use the substitution $y_i' = y_i - y_k^{n_i}$ for appropriate powers n_i.

[0]This homework is due before class on Monday March 29. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. You can ask the grader not to grade *two* of the problems.

9. (a) Let $A \subset B$ be domains and suppose that B is integral over A. Show that A is a field if and only if B is a field. (b) Let $A \subset B$ be rings and suppose that B is integral over A. Let $\mathfrak{p} \subset B$ be a prime ideal. Show that $\mathfrak{p}$ is a maximal ideal of B if and only if $\mathfrak{p} \cap A$ is a maximal ideal of A.

10. (a) Let A be an integrally closed domain with the field of fractions K. Let F/K be a Galois extension with the Galois group G. Let B be the integral closure of A in F. Show that G preserves B and that $B^G = A$. (b) Deduce from part (a) that the ring of invariants $k[x_1, \ldots, x_n]^{S_n}$ is generated by elementary symmetric functions $\sigma_1, \ldots, \sigma_n$.

11. Let A be a finitely generated k-algebra and let $B \subset A$ be a subalgebra such that A is integral over B. Show that A is a finitely generated B-module and B is a finitely generated k-algebra (Hint: consider a subalgebra of B generated by coefficients of monic equations satisfied by generators of A).

12. Let $f, g \in \mathbb{C}[x, y]$ and suppose that f is irreducible and does not divide g in $\mathbb{C}[x, y]$. (a) Show that f is still irreducible and does not divide g in $\mathbb{C}(x)[y]$. (b) Show that

$$V(f, g) = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = g(x, y) = 0\}$$

is a union of finitely many points.

13. Let $\mathfrak{p} \subset \mathbb{C}[x, y]$ be a prime ideal. Show that either $\mathfrak{p} = \{0\}$ or $\mathfrak{p}$ consists of all polynomials vanishing at some point $(a, b) \in \mathbb{C}^2$ or $\mathfrak{p} = (f)$, where $f \in \mathbb{C}[x, y]$ is an irreducible polynomial.