ALGEBRA 612, SPRING 2010. HOMEWORK 4

1. Show that S_n is solvable if and only if $n \leq 4$.
2. (a) Let $f(x) \in K[x]$ be an irreducible separable polynomial with roots

$$\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n \in \bar{K}.$$

Suppose that there exist rational functions $\theta_1(x), \ldots, \theta_n(x) \in K(x)$ such that $\alpha_i = \theta_i(\alpha)$ for any i. Suppose also that

$$\theta_i(\theta_j(\alpha)) = \theta_j(\theta_i(\alpha))$$

for any i, j. Show that $K(\alpha)/K$ is solvable in radicals. Hint: this case was examined by this famous Norwegian mathematician:

(b) Give an example of the situation as in part (a) with $K = \mathbb{Q}$ and such that the Galois group of $f(x)$ is not cyclic. Give a specific polynomial $f(x)$, and compute its roots and functions θ_i.
3. Let F/K be a finite Galois extension and let L be an intermediate subfield between F and K. Let H be the subgroup of $\mathrm{Gal}(F/K)$ mapping L to itself. Prove that H is the normalizer of $\mathrm{Gal}(F/L)$ in $\mathrm{Gal}(F/K)$.
4. Let F/K be a Galois extension with a cyclic Galois group G. Let σ be a generator of G. Show that

$$\mathrm{Ker}[\mathrm{Tr}_{F/K}] = \mathrm{Im}[\mathrm{Id}_F - \sigma].$$

In other words, if $\beta \in F$ then $\mathrm{Tr}_{F/K}(\beta) = 0$ iff $\beta = \alpha - \sigma(\alpha)$ for some $\alpha \in F$.
5. Consider the extension $\mathbb{Q} \subset F = \mathbb{Q}(\zeta_p, \sqrt[p]{2})$, where p is a prime and ζ_p is the primitive p-th root of unity. Show that $\mathrm{Gal}(F/\mathbb{Q})$ is isomorphic the semidirect product of $\mathbb{Z}/p\mathbb{Z}$ and $\mathbb{F}_p^*$.
6. Let F/K be a Galois extension with a cyclic Galois group G of order p, where $\mathrm{char}\, K = p$. Let σ be a generator of G. (a) Show that there exists $\alpha \in F$ such that $\sigma(\alpha) = \alpha + 1$. (b) Show that $F = K(\alpha)$, where α is a root of $x^p - x - a$ for some $a \in K$.
7. Suppose that $\mathrm{char}\, K = p$ and let $a \in K$. Show that the polynomial $x^p - x - a$ either splits in K or is irreducible. Show that in the latter case its Galois group is cyclic of order p.

[0]This homework is due before class on Monday March 1. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. You can ask the grader not to grade *two* of the problems. There will be an in-class midterm on Galois Theory on Wednesday, March 3.

8. Let F/K be a Galois extension with a cyclic Galois group G. Let σ be a generator of G. Let $\beta \in F$. (a) There exists $\theta \in F$ such that $\alpha \neq 0$, where

$$\alpha = \theta+\beta\sigma(\theta)+\beta\sigma(\beta)\sigma^2(\theta)+\beta\sigma(\beta)\sigma^2(\beta)\sigma^3(\theta)+\ldots+\beta\sigma(\beta)\ldots\sigma^{n-2}(\beta)\sigma^{n-1}(\theta).$$

(b) Show that $N_{F/K}(\beta) = 1$ if and only if $\beta = \alpha/\sigma(\alpha)$ for some $\alpha \in F$.

9. Let $f(x)$ be the minimal polynomial over $\mathbb{Q}$ of $\sqrt[5]{\sqrt[3]{17}+\sqrt[4]{37}}$, where all of the indicated radicals are real. Show that the splitting field of $f(x)$ is solvable over $\mathbb{Q}$.

10. Let $K = \mathbb{Q}(\zeta)$, where ζ is a primitive n-th root of unity. Show that if $n = p^r$ for some prime p then $N_{K/\mathbb{Q}}(1-\zeta) = p$.

11. Suppose that F/K and L/F are solvable extensions (recall that this does not mean that these extensions are Galois). Is it true that L/K is a solvable extension?

12. Prove that there exist infinitely many pairs of integers (a, b) such that $-4a^3 - 27b^2$ is a square in $\mathbb{Z}$.

13. Let n and m be coprime integers. Show that $\Phi_n(x)$ (the n-th cyclotomic polynomial) is irreducible over $\mathbb{Q}(\zeta_m)$.

14. Let G be a subgroup of the group of automorphisms of $\mathbb{C}(z)$ (rational functions in one variable) generated by automorphisms $z \mapsto 1-z$ and $z \mapsto 1/z$. Show that G has 6 elements and that the field of invariants $\mathbb{C}(z)^G$ is generated by one function. Find this function.

15. Let $f(x) \in K[x]$ be an irreducible polynomial of degree 5 such that its discriminant is a square in K. Find all possible Galois groups for its splitting field. For each possible Galois group, give an example of $f(x) \in \mathbb{Q}[x]$ with this Galois group.