ALGEBRA 612, SPRING 2010. HOMEWORK 2

In this set we fix a finite field extension $K \subset F$. Let $\bar{K}$ be an algebraic closure of K.

1. Let $\alpha \in F$ and let $f(x)$ be its minimal polynomial. Suppose that α is not separable over K. (a) Show that $\operatorname{char} K = p$ and $f(x) = g(x^p)$ for some polynomial $g \in K[x]$. (b) Show that there exists $k \geq 1$ such that all roots of $f(x)$ in $\bar{K}$ have multiplicity p^k and α^{p^k} is separable over K.

2. (a) Show that elements of F separable over K form a field L. We define

$$[F : K]_s := [L : K].$$

(b) Prove that the number of different inclusions of F in the algebraic closure of K over K is equal to $[F : K]_s$.

3. Show that $[F : K]_s = 1$ if and only if $\operatorname{char} K = p$ and F is generated over K by elements $\alpha_1, \ldots, \alpha_r$ such that the minimal polynomial of each α_i has the form $x^{p^{k_i}} - a_i$ for some $a_i \in K$ and a positive integer k_i.

4. Show that the primitive element theorem does not necessarily hold for finite extensions that are not separable.

5. A field k is called perfect if either $\operatorname{char} k = 0$ or $\operatorname{char} k = p$ and the Frobenius homomorphism $F : k \to k$ is an isomorphism. Show that if k is perfect then any algebraic extension of k is separable over k and perfect.

6. Let ABC be an isosceles triangle with $AB = BC$. Let AD be a bisector of the angle BAC. Suppose that (a) $AD + BD = AC$ or (b) $BD = AC$. Find the angle measure of the angle BAC in degrees. (Hint: You can either use high school geometry (but the solution will be tricky) or algebra, in which case the Law of Sines could be helpful.)

7. Let F be a splitting field of the polynomial $f \in K[x]$ of degree n. Show that $[F : K]$ divides n! (do not assume that F is separable over K).

8. Show that any element in a finite field is a sum of two squares in that field.

9. Let $F \subset \bar{K}$ be a finite Galois extension of K and let $L \subset \bar{K}$ be any finite extension of K. Consider the natural K-linear map $L \otimes_K F \to \bar{K}$. (a) Show that its image is a field, that we will denote by LF. (b) Show that LF is Galois over L. (c) Show that $\operatorname{Gal}(LF/L)$ is isomorphic to $\operatorname{Gal}(F/L \cap F)$.

10. Find the minimal polynomial over $\mathbb{Q}$ of $\sqrt[2]{3} + \sqrt[3]{3}$. Compute the Galois group of its splitting field.

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[0]This homework is due before class on Monday Feb 8. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. There is a "bail-out" provision: you can ask the grader not to grade *two* of the problems. Please indicate clearly in the beginning of your homework which problems you don't wish to be graded.

11. Let $a, b \in K$ and suppose that $f(x) = x^3 + ax + b$ has no roots in K. Let F be a splitting field of $f(x)$. Assume that $\operatorname{char} K \neq 3$. Show that

$$\operatorname{Gal}(F/K) \simeq \begin{cases} S_3 & \text{if } -4a^3 - 27b^2 \text{ is not a square in } K \\ \mathbb{Z}_3 & \text{if } -4a^3 - 27b^2 \text{ is a square in } K \end{cases}$$

12. Let $f(x) \in \mathbb{Q}[x]$ be an irreducible polynomial of prime degree p. Suppose that $f(x)$ has exactly $p-2$ real roots. Show that the Galois group of the splitting field of $f(x)$ is S_p.

13. For any $d \geq 2$, prove existence of an irreducible polynomial in $\mathbb{Q}[x]$ of degree d with exactly $d-2$ real roots (Hint: take some obvious reducible polynomial with exactly $d-2$ real roots and perturb it a little bit to make it irreducible).

14. Let G be any finite group. Show that there exist finite extensions $\mathbb{Q} \subset K \subset F$ such that F/K is a Galois extension with a Galois group G.

15. Let F be a splitting field of the polynomial $f(x) \in K[x]$. Show that $\operatorname{Gal} F/K$ acts transitively on roots of $f(x)$ if and only if $f(x)$ is irreducible (do not assume that $f(x)$ is separable).

16. Let F be a splitting field of a biquadratic polynomial $x^4 + ax^2 + b \in K[x]$. Show that $\operatorname{Gal}(F/K)$ is isomorphic to a subgroup of D_4.