# ALGEBRA 611, SPRING 2010. HOMEWORK 1 [1]

In this set we fix a field extension $K \subset F$.

**1.** Let $R$ be an infinite domain and let $f \in R[x]$. Prove that $f(r) \neq 0$ for infinitely many $r \in R$. What if $R$ is not necessarily a domain?

**2.** (a) Show $\alpha \in F$ is algebraic over $K$ if and only if $F$ contains a finite-dimensional $K$-vector subspace $L$ (not necessarily a subfield) such that

$$\alpha \cdot L \subset L.$$

(b) Find the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbb{Q}$.

**3.** (a) Let $R$ be a domain and let $R \subset K$ be its field of fractions. Show that $K$ satisfies the following universal property: for any injective homomorphism $\psi : R \to F$ to a field, there exists a unique homomorphism of fields $K \to F$ that extends $\psi$. (b) Let **Fields** be the category of fields (what can you say about morphisms in this category?). Let $R$ be a domain and let $F_R$ : **Fields** $\to$ **Sets** be a covariant functor that sends any field $k$ to the set of injective homomorphisms $R \to k$. This definition is not complete: give a complete definition of this functor and show that it is representable.

**4.** (a) Show that $f(x) = x^3 + x^2 + x + 3$ is irreducible over $\mathbb{Q}$. (b) Consider the field $F = K(\alpha)$, where $\alpha$ is a root of $f(x)$. Express $\frac{1}{2-\alpha+\alpha^2}$ as a $\mathbb{Q}$-linear combination of $1$, $\alpha$, and $\alpha^2$.

**5.** Find the degree (over $\mathbb{Q}$) of the splitting field of (a) $x^4 + x^3 + x^2 + x + 1$. (b) $x^4 - 2$.

**6.** For all positive integers $n$ and $m$, compute the degree $[\mathbb{Q}(\sqrt{n}, \sqrt{m}) : \mathbb{Q}]$.

**7.** Let $K \subset F$ be an algebraic extension and let $R$ be a subring of $F$ that contains $K$. Show that $R$ is a field.

**8.** Let $f(x) \in K[x]$ be a polynomial of degree 3. Show that if $f(x)$ has a root in a field extension $K \subset F$ of degree 2 then $f(x)$ has a root in $K$.

**9.** Let $\alpha, \beta \in F$ be algebraic over $K$, let $f(x)$ and $g(x)$ be their minimal polynomials, and suppose that $\deg f$ and $\deg g$ are coprime. Prove that $f(x)$ is irreducible in $K(\beta)[x]$.

**10.** Find the splitting field of $x^p - 1$ over $\mathbb{F}_p$.

**11.** Let $K \supset \mathbb{Q}$ be a splitting field of a cubic polynomial $f(x) \in \mathbb{Q}[x]$. Show that if $[K : \mathbb{Q}] = 3$ then $f(x)$ has 3 real roots.

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[1] This homework is due before class on Monday Feb 1. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. There is a "bail-out" provision: you can ask the grader not to grade *two* of the problems. Please indicate clearly in the beginning of your homework which problems you don't wish to be graded.

**12.** Let $\mathbb{F}_{p^n}$ be a finite field with $p^n$ elements and let $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be the Frobenius map, $F(x) = x^p$. Show that $F$ is diagonalizable (as an $\mathbb{F}_p$-linear operator) if and only if $n$ divides $p - 1$.
**13.** Let $F = K(\alpha)$ and suppose that $[F : K]$ is odd. Show that $F = K(\alpha^2)$.
**14.** Let $f(x) \in K[x]$ be an irreducible polynomial and let $g(x) \in K[x]$ be any non-constant polynomial. Let $p(x)$ be a non-constant polynomial that divided $f(g(x))$. Show that $\deg f$ divides $\deg p$.
**15.** Show that the polynomial $x^5 - t$ is irreducible over the field $\mathbb{C}(t)$ (here $t$ is a variable). Describe a splitting field.
**16.** Let $\mathbb{F}_q$ be a finite field with $q$ elements ($q$ is not necessarily prime). Compute the sum $\sum_{a \in \mathbb{F}_q} a^k$ for any integer $k$.
**17.** Show that the algebraic closure of $\mathbb{F}_p$ is equal to the union of its finite subfields:

$$\bar{\mathbb{F}}_p = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^n}.$$