ALGEBRA 611, FALL 2009. HOMEWORK 9 [1]

In this worksheet k denotes an arbitrary field and R denotes a ring (as usual, commutative and with 1).

1. Describe all prime ideals of (a) $\mathbb{Z}/24\mathbb{Z}$; (b) $\mathbb{R}[x]$.

2. (a) Show that $x^7+48x-24$ is irreducible in $\mathbb{Q}[x]$. (b) Show that $x^3+y^3+z^3$ is irreducible in $\mathbb{C}[x,y,z]$. (c) Show that $x^4+x^3+x^2+x+1$ is irreducible in $\mathbb{Q}[x]$ but reducible in $K[x]$, where $K=\mathbb{Q}\left(\frac{1+\sqrt{5}}{2}\right)$.

3. If R is a domain then 0 is the minimal prime ideal. If R is not necessarily a domain, use Zorn's lemma to show that any prime ideal of R is contained in a minimal prime ideal.

4. A ring R is called Artinian if it satisfies (dcc) for ideals, i.e. any descending chain of ideals stabilizes. (a) Show that R is Artinian if and only if any set of ideals contains a minimal element. (b) Show that any finite ring is Artinian. (c) Show that an Artinian domain is a field.

5. (a) Show that a subgroup of $\mathbb{Q}/\mathbb{Z}$ generated by $\sqrt{5}$ is dense (with respect to the usual topology on $\mathbb{Q}/\mathbb{Z}$). (b) Prove that a subring of $k[x,y]$ generated by all monomials x^ny^m such that $n/m<\sqrt{5}$ is not Noetherian.

6. Let S be a subset of lattice points (i.e. points with integral coordinates) in the first quadrant. Suppose that S satisfies the following property: if $(n,m)\in S$ then $(n+1,m)\in S$ and $(n,m+1)\in S$. Prove that S is a union of finitely many shifted quadrants, i.e. subsets of the form $(n+a,m+b)$, where (n,m) is fixed and a,b are arbitrary nonnegative integers.

7. Prove that the ring $k[[x,y]]$ of power series in two variables is Noetherian (Hint: imitate the proof of Hilbert's basis theorem).

8. Prove that if R is a UFD then the ring $R[[x]]$ is a UFD.

9. This problem is a bit off-topic, but fun: (a) Start with any $x\equiv 1 \mod 4$ but $x\not\equiv 1 \mod 8$. Show that $x^{2^r}\equiv 1 \mod 2^{r+2}$ but $x^{2^r}\not\equiv 1 \mod 2^{r+3}$. (b) Show that the multiplicative group $(\mathbb{Z}/2^n\mathbb{Z})^*$ is generated by 5 and -1, i.e. any odd number is equal to $\pm 5^s$ (for some $s\geq 0$) modulo 2^r.

If you think you can make money on this problem, you are right, but you will have to wait until 2017 when the US Patent 5923888 expires.

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[1] This homework is due before class on Friday Dec 11. The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. There is a "bail-out" provision: you can ask the grader not to grade *two* of the problems. Please indicate clearly in the beginning of your homework which problems you don't wish to be graded.