ALGEBRA 611, FALL 2009. HOMEWORK 1

This homework is due before the class on Monday September 21. These problems will be discussed during the review section on Monday at 4pm.

The grader will grade 5 random problems from this assignment. A problem with multiple parts (a), (b), etc. counts as one problem. Please make sure that all solutions are complete and accurately written.

There is a "bail-out" provision: you can ask the grader not to grade *two* of the problems. Please indicate clearly in the beginning of your homework which problems you don't wish to be graded.

Do not assume that vector spaces are finite-dimensional unless otherwise stated. For infinite-dimensional problems, the linear Hahn-Banach theorem (see below) is very useful.

1. Prove that any matrix of rank one is a product of a row vector and a column vector.

2. Let A and B be $k \times n$ matrices. Show that $\operatorname{rk} A = \operatorname{rk} B$ if and only if $A = CBD$ for some invertible matrices C, D.

3. Let A be an $n \times k$ matrix and let B be a $k \times m$ matrix. Show that

$$\operatorname{rk}(AB) \leq \min(\operatorname{rk} A, \operatorname{rk} B).$$

Is it true that two sides are always equal?

4. Prove that if Γ and Γ' are two bases of $\mathbb{F}^n$ then for any $e \in \Gamma \setminus \Gamma'$ there exists $f \in \Gamma' \setminus \Gamma$ such that $\Gamma \setminus \{e\} \cup \{f\}$ is a basis of $\mathbb{F}^n$.

5. Consider the matrix of real numbers

$$\begin{bmatrix} 1 & -4 & 3 & 1 & 0 \\ -2 & 1 & 1 & 2 & -1 \\ 2 & 3 & -1 & 1 & 2 \\ 1 & 0 & 3 & 4 & 1 \end{bmatrix}$$

Use row reduction to (a) Find a basis of the row space. (b) Find a basis of the column space. (c) Find a basis of the null space. (d) Extend your solution for (a) to a basis of $\mathbb{R}^5$. (e) Shrink the set of column vectors to a basis of their linear span.

6. Let $L : \mathbb{F}^n \to \mathbb{F}^k$ be a linear map. Prove that one can choose bases of $\mathbb{F}^n$ and $\mathbb{F}^k$ in such a way that

$$[L] = \begin{bmatrix} I_p & 0_{p,n-p} \\ 0_{k-p,p} & 0_{k-p,n-p} \end{bmatrix}$$

where I_p is a $p \times p$ identity matrix and $0_{m,l}$ is an $m \times l$ matrix of zeros.

7. Let A be an $n \times n$ matrix such that $A^2 = A$. Prove that $\operatorname{Ker} A \oplus \operatorname{Im} A = \mathbb{F}^n$.

8. (Linear Hahn-Banach Theorem). Let V be a vector space, let $U \subset V$ be a vector subspace, and let $v \in V \setminus U$ be a vector. Prove that there exists a linear function $f \in V^*$ such that $f|_U = 0$ and $f(v) = 1$.

9. For any vector subspace $U \subset V$, let

$$\mathrm{Ann}(U) = \{f \in V^* \mid f|_U = 0\}$$

be its *annihilator*. Prove that (a) if V is finite-dimensional then $\mathrm{Ann}(\mathrm{Ann}(U))$ is canonically isomorphic to U; (b) $(V/U)^*$ is canonically isomorphic to $\mathrm{Ann}(U)$. (To show that two vector spaces are canonically isomorphic, you have to construct an isomorphism between them that does not depend on choices of bases.)

10. Prove that if V is finite-dimensional then $\dim \mathrm{Ann}\, U + \dim U = \dim V$.

11. Let $L : U \to V$ be a linear map. Prove that $\mathrm{Ann}(\mathrm{Ker}\, L) = \mathrm{Im}(L^*)$.

12. Let $U_1, U_2, U_3 \subset V$ be vector subspaces. (a) Give necessary and sufficient conditions when $U_1 \cup U_2$ is a vector subspace. (b) Prove or give a counterexample $U_1 \cap (U_2 + U_3) = U_1 \cap U_2 + U_1 \cap U_3$.

13. Let q_i for $i \geq 1$ be a linear functional on $\mathbb{F}[x]$ defined as follows: $q_i(p) = p(i)$ for any polynomial $p \in \mathbb{F}[x]$. (a) Prove that q_i's are linearly independent. (b) Prove that q_i's do not span the dual vector space $\mathbb{F}[x]^*$.

14. Let A be a 3×3 matrix with integral entries. Prove that if all 2×2 minors of A are divisible by 9 then $\det A$ is divisible by 27.