Name and section: ______________________

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1. **(20 points)** Let $f(x) = e^x + x + 1$.
   (a) Find the **second-order** Taylor polynomial $P_2(x)$ for $f(x)$ about $x_0 = 0$.
   (b) Assume that $P_2(x)$ is used in order to approximate $f(x)$ on $[0, 1]$. Then, upon using the **remainder term** R_2, find an **upper bound** for

$$\max_{x\in[0,1]} |f(x) - P_2(x)|.$$

Solution:

(a) From the function given, the $P_2(x)$ reads

$$P_2(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2!}f''(x_0)(x-x_0)^2 \Rightarrow \boxed{P_2(x) = 2 + 2x + \frac{1}{2}x^2}.$$

(b) Next, using the remainder term $R_2(x)$ we obtain

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x - x_0)^3 \Rightarrow \boxed{R_2(x) = \frac{e^\xi}{6}x^3}.$$

Note that we are interested in finding both ξ and x in $[0, 1]$ such that $|R_2|$ is maximized. Thus, we have

$$|R_2(x)| = \left|\frac{e^\xi}{6}x^3\right| \le \left|\frac{e^1}{6}1^3\right| \Rightarrow \boxed{|R_2(x)| \lesssim 0.453}.$$

2. **(20 points)** Let $f(x) = x^2 - 3$ with $a = 1$ and $b = 2$. Furthermore, assume that we want to use the **bisection method** to find the root of $\sqrt{3} \approx 1.732050807568877$. Note also, that the initial interval is $[a_0, b_0] = [1, 2]$.
   (a) Is the method guaranteed to converge to the root? Explain why.
   (b) Perform **2 bisections** by showing your steps. What is the value of a_2 if the root is in the interval $[a_2, b_2]$?

Solution:

(a) To determine whether bisection converges to the root, we have to find out whether the criteria of the Intermediate Value Theorem are fulfiled, that is, if the function f changes sign. To this end, we have $f(1)f(2) = -2 < 0$, so indeed the bisection method will converge. Note also, that the root given lies in $[1, 2]$.

(b) In the first step, we compute the midpoint denoted by p of $[a_0, b_0] = [1, 2]$, which has the value of $p = (a_0 + b_0)/2 = 3/2$. Subsequently, we check $f(1)f(3/2) = 3/2 > 0$ and $f(3/2)f(2) = -3/4 < 0$. Therefore, $[a_1, b_1] = [3/2, 2]$. Next, we find the new p which is given by $p = (a_1 + b_1)/2 = 7/4$. We must check now whether the function $f(x)$ changes sign in the intervals $[3/2, 7/4]$ or $[7/4, 2]$. To this end we have $f(3/2)f(7/4) = -3/64 < 0$ and $f(7/4)f(2) = 1/16 > 0$. Thus, we obtain $[a_2, b_2] = [3/2, 7/4]$, or $a_2 = 3/2$ in question. Note also that $x^* \in [a_2, b_2]$ still.

3. **(30 points)** Suppose that we want to use **fixed point iterations** of the form of $x_{k+1} = g(x_k)$ with $k = 0, 1, \ldots,$ in order to find the value of x such that $x^2 - 2x = 3$. Note that $g(x)$ stands for the **iteration function**.

(a) Find the two roots, namely, x_1^* and x_2^*, of the underlying quadratic equation analytically.

(b) Determine **three** iteration functions $g(x)$. Then, which choice of yours can be used to find the **positive** root of the previous step? Explain why.

(c) What is the **order of convergence** in this case? Justify your answer.

Solution:

(a) At first, the equation given is a quadratic one and can be written as $x^2 - 2x - 3 = 0$ with roots $x_1^* = -1$ and $x_2^* = 3$ obtained by known formulas.

(b) The three iteration functions can be obtain as follows

$$x^2 - 2x = 3 \quad \Rightarrow \quad x = \frac{1}{2}\left(x^2 - 3\right) \quad \text{with} \quad g(x) = \frac{1}{2}\left(x^2 - 3\right),$$
$$x^2 - 2x = 3 \quad \Rightarrow \quad x = \pm\sqrt{2x + 3} \quad \text{with} \quad g(x) = \pm\sqrt{2x + 3}.$$

Next, and based on the functions $g(x)$ obtained above, we must examine whether $|g'(x_2^*)| < 1$ holds. In particular, we have

$$\begin{aligned} |g'(x)| &= |x| \Rightarrow |g'(x_2^*)| = 3 > 1, \\ |g'(x)| &= \left|\frac{1}{\sqrt{2x + 3}}\right| \Rightarrow |g'(x_2^*)| = 1/3 < 1, \end{aligned}$$

respectively. This way, we conclude that the positive root $x_2^* = 3$ can be found by using $g(x) = \sqrt{2x + 3}$ (with the plus sign!).

(c) The fixed point iteration will converge **linearly** since $|g'(x_2^*)| \neq 0$. This can be immediately seen by the Taylor expansion of the iteration function $g(x)$ about x_2^*

$$\begin{aligned}
g(x) &= g(x_2^*) + g'(x_2^*)(x - x_2^*) + \frac{g''(\xi)}{2}(x - x_2^*)^2 \xrightarrow{(x=x_k)} \\
x_{k+1} &= x_2^* + g'(x_2^*)(x_k - x_2^*) + \frac{g''(\xi)}{2}(x_k - x_2^*)^2 \Rightarrow \\
\frac{x_{k+1} - x_2^*}{x_k - x_2^*} &= g'(x_2^*) + \frac{g''(\xi)}{2}(x_k - x_2^*) \Rightarrow \\
\lim_{k\to\infty} \frac{|x_{k+1} - x_2^*|}{|x_k - x_2^*|} &= |g'(x_2^*)|
\end{aligned}$$

Note that the convergence factor ρ and rate of convergence α are $\rho = |g'(x_2^*)| = 1/3 < 1$ and $\alpha = 1$, respectively.

4. **(30 points)** Assume that we wish to find x such that $e^x + x = 3$.
   (a) Reformulate the problem as $f(x) = 0$ by identifying the function $f(x)$.
   (b) What is the iteration function $g(x)$ for **Newton's method** applied to $f(x)$?
   (c) Perform just **one** step of Newton's method with initial guess $x_0 = 0$.
   (d) Of course, if we keep iterating we will converge to the actual root x^*. What would be the **order of convergence** of the iterates? Justify your answer.

Solution:

(a) Obviously, $f(x) = 0$ with $f(x) = e^x + x - 3$.

(b) From Newton's method, we know that

$$g(x) = x - \frac{f(x)}{f'(x)} \Rightarrow \boxed{g(x) = x - \frac{e^x + x - 3}{e^x + 1}}.$$

(c) As per Newton's method

$$x_{k+1} = x_k - \frac{e^{x_k} + x_k - 3}{e^{x_k} + 1},$$

thus, at $k = 0$ we have that

$$x_1 = x_0 - \frac{e^{x_0} + x_0 - 3}{e^{x_0} + 1} \Rightarrow \boxed{x_1 = 1}.$$

(d) To obtain the order of convergence of Newton's method, we know that

$$g'(x) = 1 - \frac{f'^2(x) - f(x)f''(x)}{f'^2(x)} \Rightarrow g'(x) = \frac{f(x)f''(x)}{f'^2(x)} \xrightarrow{x=x^*} \boxed{g'(x^*) = 0},$$

since $f(x^*) = 0$ with x^* denoting the fixed point. Furthermore, recall Taylor's expansion of the iteration function $g(x)$ about x^*

$$
\begin{aligned}
&g(x) = g(x^*) + \cancelto{0}{g'(x^*)}(x - x^*) + \frac{g''(x^*)}{2}(x - x^*)^2 + \frac{g'''(\xi)}{6}(x - x^*)^3 \xrightarrow{(x=x_k)} \\
&x_{k+1} = x^* + \frac{g''(x^*)}{2}(x_k - x^*)^2 + \frac{g'''(\xi)}{6}(x_k - x^*)^3 \Rightarrow \\
&\frac{x_{k+1} - x^*}{(x_k - x^*)^2} = \frac{1}{2} g''(x^*) + \frac{g'''(\xi)}{6}(x_k - x^*) \Rightarrow \\
&\lim_{k\to\infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} = \frac{1}{2}|g''(x^*)|,
\end{aligned}
$$

where the convergence factor ρ and rate of convergence α are given by $\rho = |g''(x^*)|/2$ and $\alpha = 2$, respectively. Finally, the fact that $\alpha = 2$ signals that the convergence to x^* in Newton's method is **quadratic** provided that $g''(x^*) \neq 0$.

Duration: 1h 15 min

Good luck!