Name and section: ____________________

ID number: ____________________

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1. (15 points) Let $f(x) = \cos x + x$.
   (a) (7 points) Find the **second-order** Taylor polynomial $P_2(x)$ for $f(x)$ about $x_0 = 0$.
   (b) (8 points) Assume that $P_2(x)$ is used in order to approximate $f(x)$ on $[0, \pi/2]$. Then, upon using the **remainder term** R_2, find an **upper bound** for

$$\max_{x\in[0,\pi/2]} |f(x) - P_2(x)|.$$

Solution:

(a) From the function given, the $P_2(x)$ reads

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 \Rightarrow \boxed{P_2(x) = 1 + x - \frac{1}{2}x^2}.$$

(b) Next, using the remainder term $R_2(x)$ we obtain

$$R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x - x_0)^3 \Rightarrow \boxed{R_2(x) = \frac{\sin \xi}{6} x^3}.$$

Note that we are interested in finding both ξ and x in $[0, \pi/2]$ such that $|R_2|$ is maximized. Thus, we have

$$|R_2(x)| = \left| \frac{\sin \xi}{6} x^3 \right| \leq \left| \frac{\sin(\pi/2)}{6} \left(\frac{\pi}{2}\right)^3 \right| = \frac{\pi^3}{48} \Rightarrow \boxed{|R_2(x)| \lesssim 0.645964}.$$

2. (20 points) Let $f(x) = x^2 - 7$ with $a = 2$ and $b = 3$. Furthermore, assume that we want to use the **bisection method** to find the root of $\sqrt{7} \approx 2.6457513110645907$. Note also, that the initial interval is $[a_0, b_0] = [2, 3]$.
   (a) (5 Points) Is the method guaranteed to converge to the root? **Explain why**.
   (b) (15 Points) Perform **2 bisections** by showing your steps. What is the value of a_2 if the root is in the interval $[a_2, b_2]$?

Solution:

(a) To determine whether bisection converges to the root, we have to find out whether the criteria of the Intermediate Value Theorem are fulfiled, that is, if the function f changes sign and is continuous. To this end, we have $f(2)f(3) = -6 < 0$, so indeed the bisection method will converge. Note also, that the root given lies in $[2, 3]$.

(b) In the first step, we compute the midpoint denoted by p of $[a_0, b_0] = [2, 3]$, which has the value of $p = (a_0 + b_0)/2 = 5/2$. Subsequently, we check $f(2)f(5/2) = 9/4 > 0$ and $f(5/2)f(3) = -3/2 < 0$. Therefore, $[a_1, b_1] = [5/2, 3]$. Next, we find the new p which is given by $p = (a_1 + b_1)/2 = 11/4$. We must check now whether the function $f(x)$ changes sign in the intervals $[5/2, 11/4]$ or $[11/4, 3]$. To this end we have $f(5/2)f(11/4) = -27/64 < 0$ and $f(11/4)f(3) = 9/8 > 0$. Thus, we obtain $[a_2, b_2] = [5/2, 11/4]$, or $a_2 = 5/2$ in question. Note also that $x^* \in [a_2, b_2]$ still.

3. (20 points) Suppose that we want to use **fixed point iterations** of the form of $x_{k+1} = g(x_k)$ with $k = 0, 1, \ldots$, in order to find the value of x such that $x^2 + 4x = 5$. Note that $g(x)$ stands for the **iteration function**.
   (a) (5 points) Find the two roots, namely, x_1^* and x_2^*, of the underlying quadratic equation **analytically**.
   (b) (7 points) Determine **three** iteration functions $g(x)$. Then, which choice of yours can be used to find the **negative** root of the previous step? **Explain why**.
   (c) (8 points) What is the **order of convergence** in this case? Justify your answer.

Solution:

(a) At first, the equation given is a quadratic one and can be written as $x^2 + 4x - 5 = 0$ with roots $x_1^* = -5$ and $x_2^* = 1$ obtained by known formulas.

(b) The three iteration functions can be obtain as follows

$$x^2 + 4x = 5 \;\Rightarrow\; x = \frac{1}{4}\left(5 - x^2\right) \quad \text{with} \quad g(x) = \frac{1}{4}\left(5 - x^2\right),$$
$$x^2 + 4x = 5 \;\Rightarrow\; x = \pm\sqrt{5 - 4x} \quad \text{with} \quad g(x) = \pm\sqrt{5 - 4x}.$$

Note that in the last two choices $5 - 4x \geq 0 \Rightarrow x \leq 5/4$. Next, and based on the functions $g(x)$ obtained above, we must examine whether $|g'(x_1^*)| < 1$ holds. In particular, we have

$$|g'(x)| \;=\; \left|-\frac{x}{2}\right| \Rightarrow |g'(x_1^*)| = \frac{5}{2} > 1,$$
$$|g'(x)| \;=\; \left|-\frac{2}{\sqrt{5 - 4x}}\right| \Rightarrow |g'(x_1^*)| = 2/5 < 1,$$

respectively. This way, we conclude that the negative root $x_1^* = -5$ can be found by using $g(x) = -\sqrt{5 - 4x}$ (with the minus sign!).

(c) The fixed point iteration will converge **linearly** since $|g'(x_1^*)| \neq 0$. This can be immediately seen by the Taylor expansion of the iteration function $g(x)$ about x_1^*

$$\begin{aligned}
g(x) &= g(x_1^*) + g'(x_1^*)(x - x_1^*) + \frac{1}{2}(x - x_1^*)^2 g''(\xi) \xrightarrow{(x=x_k)} \\
g(x_k) &= g(x_1^*) + g'(x_1^*)(x_k - x_1^*) + \frac{1}{2}\left(x_k - x_1^*\right)^2 g''(\xi) \\
x_{k+1} &= x_1^* + g'(x_1^*)(x_k - x_1^*) + \frac{1}{2}(x_k - x_1^*)^2 g''(\xi) \Rightarrow \\
\frac{x_{k+1} - x_1^*}{x_k - x_1^*} &= g'(x_1^*) + \frac{1}{2}(x_k - x_1^*) g''(\xi) \Rightarrow \left(\lim_{k\to\infty} x_k = x^*\right) \\
\lim_{k\to\infty} \frac{|x_{k+1} - x_1^*|}{|x_k - x_1^*|} &= |g'(x_1^*)|
\end{aligned}$$

Note that the convergence factor ρ and rate of convergence α are $\rho = |g'(x_1^*)| = 2/5 < 1$ and $\alpha = 1$, respectively.

4. (20 points) Assume that a function f is three times differentiable, that is, $f \in C^3[a,b]$ and there is a root $x^* \in [a,b]$ such that $f(x^*) = 0$ and $f'(x^*) \neq 0$. Show that Newton's method converges **quadratically**.

Solution:

(a) **One approach:**

Let us first define the **absolute error** as $e_k \doteq |x_k - x^*|$. Then, consider the Taylor expansion for $g(x)$ about x^*

$$g(x) = g(x^*) + g'(x^*)(x - x^*) + \frac{1}{2}(x - x^*)^2 g''(\xi), \tag{1}$$

where

$$\begin{aligned}
g'(x) &= \frac{f''(x) f(x)}{[f'(x)]^2}, \\
g''(x) &= \frac{f'(x) f''(x) + f(x) f'''(x)}{[f'(x)]^2} - \frac{2 f(x)\,[f''(x)]^2}{[f'(x)]^3}.
\end{aligned}$$

Since $f(x^*) = 0$ and $f'(x^*) \neq 0$, then

$$g'(x^*) = 0, \tag{2}$$

$$g''(x^*) = \frac{f''(x^*)}{f'(x^*)}. \tag{3}$$

Inserting Eqs. (2) and (3) into Eq. (1), we obtain at $x = x_k$

$$\begin{aligned}
g(x_k) &= g(x^*) + \frac{1}{2}(x_k - x^*)^2 g''(\xi) \Rightarrow \\
x_{k+1} &= x^* + \frac{1}{2}(x_k - x^*)^2 g''(\xi) \Rightarrow \\
x_{k+1} - x^* &= \frac{1}{2}(x_k - x^*)^2 g''(\xi) \Rightarrow \\
\underbrace{|x_{k+1} - x^*|}_{e_{k+1}} &= \frac{1}{2}\underbrace{|x_k - x^*|^2}_{e_k^2}|g''(\xi)| \Rightarrow \\
e_{k+1} &= \frac{|g''(\xi)|}{2}e_k^2.
\end{aligned}$$

The $g''(x)$ is a continuous function since $g(x)$ is a smooth one. Hence, there exists a constant $M \in \mathbf{R}$ such that

$$\frac{|g''(\xi)|}{2} \leq M,$$

for all points ξ sufficiently close to x^*. Then, the previous inequality yields

$$\boxed{e_{k+1} \leq M e_k^2},$$

proving the quadratic convergence of Newton's method.

(b) **Second approach:**

Let us Taylor expand the iteration function $g(x)$ about x^*

$$\begin{aligned}
g(x) &= g(x^*) + g'(x^*)(x - x^*) + \frac{1}{2}(x - x^*)^2 g''(\xi) \xrightarrow{(x=x_k)} \\
g(x_k) &= g(x^*) + \frac{1}{2}(x_k - x^*)^2 g''(\xi) \Rightarrow \\
x_{k+1} &= x^* + \frac{1}{2}(x_k - x^*)^2 g''(\xi) \Rightarrow \\
x_{k+1} - x^* &= \frac{1}{2}(x_k - x^*)^2 g''(\xi) \Rightarrow \\
\frac{x_{k+1} - x^*}{(x_k - x^*)^2} &= \frac{1}{2}g''(\xi) \Rightarrow \\
\frac{|x_{k+1} - x^*|}{|x_k - x^*|^2} &= \frac{1}{2}|g''(\xi)|,
\end{aligned}$$

which implies that $\alpha = 2$ with convergence factor $\rho = \frac{1}{2}|g''(\xi)|$. Note that the latter is based on the definition on convergence of series that we discussed in class.

5. (25 points) Consider the following system of **two** nonlinear equations

$$y - x^3 = 0,$$

$$x^2 + y^2 = 1.$$

(a) (5 points) Write down the associated **Jacobian matrix** for this system.

(b) (20 points) Perform **one** step of Newton's method with initial guesses $x_0 = 1$ and $y_0 = 2$.

The inverse of a 2×2 *matrix* A *of the form of*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution:

(a) At first, we have $f(x, y) = y - x^3$ and $g(x, y) = x^2 + y^2 - 1$. The Jacobian matrix J is given by

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \Rightarrow \boxed{J = \begin{bmatrix} -3x^2 & 1 \\ 2x & 2y \end{bmatrix}}.$$

(b) Next, we obtain the inverse of the Jacobian matrix using the formula provided

$$J^{-1} = -\frac{1}{6x^2y + 2x} \begin{bmatrix} 2y & -1 \\ -2x & -3x^2 \end{bmatrix}.$$

This way, Newton's method follows

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ y_k \end{bmatrix} + \frac{1}{6x^2y + 2x} \begin{bmatrix} 2y_k & -1 \\ -2x_k & -3x_k^2 \end{bmatrix} \begin{bmatrix} y_k - x_k^3 \\ x_k^2 + y_k^2 - 1 \end{bmatrix},$$

and applying it at $k = 0$, we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{14} \begin{bmatrix} 4 & -1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \Rightarrow \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{14} \begin{bmatrix} 0 \\ -14 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

Duration: 1h 30 min

Good luck!