Midterm SOLUTIONS

March 12, 2009

Instructions: Show all your work for full credit, and box your answers when appropriate. Unless otherwise noted, all rates are per annum with continuous compounding.

1. (a) Let $Y_t = B_t^3 + t^2$ where B_t is Brownian motion. Compute dX.

   Solution:

   Use Ito's Lemma with $X_t = B_t$, $G(x,t) = x^3 + t^2$. So $dX = 0 * dt + 1 * dB, Y_t = G(X_t, t)$, and

$$\begin{aligned} dY &= \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x} * 0 + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} * 1\right) + \frac{\partial G}{\partial x} * 1dB \\ &= (2t + \frac{1}{2}6B)dt + 3B^2 dB \end{aligned}$$

   (b) Suppose Y follows geometric brownian motion, show that Y^3 also follows geometric Brownian motion.

   Solution: Use Ito's Lemma with $G(y,t) = y^3$. So $Y_t^3 = G(Y_t, t)$

$$\begin{aligned} d(Y^3) &= \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial y} * (\mu Y) + \frac{1}{2}\frac{\partial^2 G}{\partial y^2} * (\sigma Y)\right) + \frac{\partial G}{\partial y}(\sigma Y)dB \\ &= (0 + 3Y^2 * (\mu Y) + \frac{1}{2}6Y * (\sigma Y))dt + 3Y^2 * (\sigma Y)dB \\ &= Y^3(3\mu + 3\sigma)dt + Y^3(3\sigma)dB \end{aligned}$$

   This is GBM with drift $(3\mu + 3\sigma)$ and volatility 3σ.

2. IBM is currently trading at \$80. Traders believe it follows the log-normal model: its volatility is 25%, and it is expected to grow at a rate of 6%. (That is, the expected return of the stock is 6% per annum, with continuous compounding.) The risk-free rate is 5%. IBM will pay a dividend of \$10 in 2 months. Consider both a European and American call option on IBM with strike \$70 and expiring in 3 months. Compute the Black-Scholes price of the European call option, and Black's approximation for the American call option price.

   Solution:

   For the European call, first we compute the effective stock price

$$S_0^e ff = 80 - 10e^{-0.05*2/12} = 70.08.$$

We plug this into d_1 and d_2:

$$d_1 = \frac{\ln(70.08/80) + (0.05 + .25^2/2) * 3/12}{.25 * \sqrt{3/12}} = 0.1716, \quad d_2 = d_1 - .25 * \sqrt{3/12} = .0466.$$

and then into the Black-Scholes formula

$$c = 70.08 F_Z(0.1716) - 70 e^{-0.05*3/12} F_Z(.0466) = 3.96$$

For the American call, Black's approximation tells us to compute price of a European call which expires just before the dividend is paid. In this case our expiration $T = 2/12$ and our "effective stock price " is 80 since no dividend is missed.

$$d_1 = \frac{\ln(80/80) + (0.05 + .25^2/2) * 2/12}{.25 * \sqrt{2/12}} = 1.441, \quad d_2 = d_1 - .25 * \sqrt{2/12} = 1.339$$

$$c = 80(F_Z(1.441) - 70 e^{-0.05*2/12} F_Z(1.339) = 10.85$$

The approximation for the American price is the greater of these two European prices, which is 10.85.

3. In these uncertain times, some investors like to purchase the Inverse Derivative, which pays you the multiplicative inverse of (that is, "one over") a stock price at some future specified maturity date. Google is currently trading at \$300. Traders believe it follows the log-normal model: its volatility is 20%, and it is expected to grow at a rate of 10%. The risk-free rate is 5%. Compute the price of the Inverse Derivative on Google, maturing in 6 months.

   Solution: Let Z be an $N(0,1)$ random variable. The price is the expected discounted payoff in the risk neutral world (note that μ is irrelevant)

$$\begin{aligned} e^{-rT} E[S_T^{-1}] &= e^{-rT} E[S_0^{-1} e^{-(r-\sigma^2/2)T - \sigma\sqrt{T}Z}] \\ &= e^{-rT} S_0^{-1} e^{-(r-\sigma^2/2)T} E[e^{-\sigma\sqrt{T}Z}] \\ &= e^{-rT} S_0^{-1} e^{-(r-\sigma^2/2)T} e^{\sigma^2 T/2} \\ &= 0.0032348184 \end{aligned}$$

4. Assume the price of a stock S_t is log-normal:

$$dS = \mu S dt + \sigma S dB$$

   where B_t is Brownian motion, the constant μ is the expected growth, and the constant σ is the volatility. The risk-free rate is r. Today is time 0. Compute today's price of the so-called "Digital Option," which pays you \$1 at time T if the stock's actual (realized) growth rate beats the risk-free rate over the time period

$[0, T]$ and \$0 if it does not. In other words, the derivative holder gets \$1 if a given investment today in the stock beats the same investment in the risk-free bond over the time period $[0, T]$, and \$0 if it does not.

Solution: Here we need to use the probability trick that is X is a random variable which is 1 if event A occurs and 0 otherwise, then $E[X] = P(A)$, the probability of event A occurring. Let X be the pay-off of this digital option. The price is the expected discounted payoff in the risk neutral world (note that μ is irrelevant)

$$
\begin{aligned}
e^{-rT}E[X] &= e^{-rT}P(S_T > S_0 e^{rT}) \\
&= e^{-rT}P(S_0 e^{(r-\sigma^2/2)T+\sigma\sqrt{T}Z} > S_0 e^{rT}) \\
&= e^{-rT}P(e^{\sigma\sqrt{T}Z} > e^{\sigma^2 T/2}) \\
&= e^{-rT}P(Z > \frac{\sigma^2 T/2}{\sigma\sqrt{T}}) \\
&= e^{-rT}(1 - P(Z < \frac{\sigma^2 T/2}{\sigma\sqrt{T}}))
\end{aligned}
$$

$$= e^{-rT}(1 - F_Z(\frac{\sigma\sqrt{T}}{2}))$$