Math 441 **Mathematics of Finance** **Spring 2013**

Midterm
March 05, 2013

Name: SOLUTION KEY

Instructions: Show all your work for full credit, and box your answers when appropriate. Unless otherwise noted, the risk free rate is per annum with continuous compounding and stocks do not pay dividends.

1. The current price of Microsoft (MSFT) is \$30. In the next year, it has a 50% chance of going up by 15% and a 50% chance of going down by 10%. Let X denote the payoff of a one-year European put on MSFT with strike \$31. Compute the variance of the payoff, $V(X)$.

   Solution: X has a 50% chance of equalling $\max(0, 31 - 33) = 0$ and a 50% chance of equalling $\max(0, 31 - 27) = 4$. The expectation is

   $$E[X] = 0.5 \times 0 + 0.5 \times 4 = 2$$

   So the variance is

   $$V(X) = 0.5 \times 0^2 + 0.5 \times 4^2 - 2^2 = 4.$$

2. The current price of IBM (IBM) is \$200. Assume the risk-free rate is 0%. A 3-month European call with strike 210 trades for \$8. Find the range of prices of IBM in 3 months for which the 3-month IBM (European) straddle with strike 210 has a positive profit.

   Solution: Let $S_{0.25}$ denote IBM price in 3 months. Let $c = 8$ and p denote the price of the 3-month call and put. Put-call parity says

   $$p = c + Ke^{-rT} - S_0 = 8 + 210e^0 - 200 = 18.$$

   The profit is the payoff minus the future value of the cost which in this case is just payoff minus cost since the risk-free rate is 0.

   $$\text{Profit} = \max(S_{0.25} - 210, 0) + \max(210 - S_{0.25}, 0) - (8 + 18) = |S_{0.25} - 210| - 26.$$

   This is positive for $|S_{0.25} - 210| - 26 > 0$, that is $S_{0.25} < 210 - 28 = 186$ or $S_{0.25} > 210 + 26 = 236$.

3. The “half-off” is an option, which if exercised, allows the holder at expiration to purchase the underlying asset for half the market price. Facebook (FB) currently trades for \$30. Every 3 month FB can increase by 10% or decrease by 10%. Compute the price of a 6-month half-off on FB. The risk-free rate is 3%.

Solution: The two step binomial tree has nodes at expiration time $t = 6/12$ of

$$S_0uu = 30(1.1)^2 = 36.3, \quad S_0du = 30(0.9)(1.1) = 29.7, \quad S_0dd = 30(0.9)^2 = 24.3.$$

The pay-off of this derivative (which is always exercised) is

$$h_{uu} = 36.3 - \frac{36.3}{2} = 18.15, \quad h_{ud} = 29.7/2 = 14.85, \quad h_{dd} = 24.3/2 = 12.15$$

The risk-neutral up-probability is

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.03\times 0.25} - 0.9}{1.1 - 0.9} = 0.5376$$

The price of the half-off is the discounted expected risk-neutral payoff

$$h = e^{-0.03\times 0.25}((0.5376)^2(18.15)+(2)(0.5376)(0.4624)(14.85)+(0.4624)^2(12.15)) = 15.00.$$

Super-quick Solution: No one got this, but one could argue as follows. The payoff of the half-off at time $t = 6/12$, regardless of whether AAPL increases or decreases, is

$$S_{\frac{6}{12}} - \frac{S_{\frac{6}{12}}}{2} = \frac{S_{\frac{6}{12}}}{2}$$

This means that the half-off is equivalent to half a stock. (In the notation of the class, we build a risk-free portfolio by a short position in the half-off and long $N = 1/2$ shares of AAPL.) The price of half a stock today is $S_0/2 = 30/2 = 15$. Note that nothing is assumed about a binomial tree; this argument works regardless of the stock model.

4. Apple (AAPL) currently trades for \$500. In one year, it will either trade for 450, 500 or 550. In the risk-neutral world, the probabilities of 450, 500, or 550 are 10%, 50% and 40%, respectively. In the real world, the probabilities of 450, 500, or 550 are 20%, 25% and 55%, respectively.

   (a) According to investors’ attitudes on AAPL, is the world risk-averse or risk-prone? Justify your answer with some numerical computations.

   Solution: Compare AAPL’s expected return r in the RNW with its expected return μ in the RW. Let S_1 be the price of AAPL in one year.

$$\begin{aligned} E_{RNW}[S_1] &= 0.1 \times 450 + 0.5 \times 500 + 0.4 \times 550 = 515 = 500e^{r} \\ E_{RW}[S_1] &= 0.2 \times 450 + 0.25 \times 500 + 0.55 \times 550 = 517.50 = 500e^{\mu} \end{aligned}$$

Without computing r and μ explicitly, we see that $\mu > r$. Since the expected return in the real world is higher than that in the risk-neutral world, we see the world is risk-averse.

(b) Can one deduce the risk-free rate from this problem? If so, compute it. If not, explain why one cannot.

Solution: Yes, the risk-free rate is the expected return of AAPL in the RNW

$$515 = E_{RNW}[S_1] = S_0 e^r = 500e^r \longrightarrow r = \ln(515/500) = 0.296 = 2.96\%.$$

5. Google (GOOG) is currently trading for \$600. The stock does not pay dividends. The risk-free rate is 3%. A 6-month at-the-money GOOG American put is trading for \$25 more than at the 6-month at-the-money GOOG American call. Construct an arbitrage opportunity involving one of each of these options. What is the minimum guaranteed risk-free profit of this strategy in today's dollars?

   Solution: The American version of put-call "somewhat" parity is $K \geq P - C + S_0 \geq Ke^{rT}$. Since $K = 600$ and $P - C + S_0 = 25 + 600 > 600$ we see the first inequality is violated. So short the put and the asset, long the call, deposit the $+P + S_0 - C = 625$ in a bank. Recover the stock (so that your asset position is $-1 + 1 = 0$) by waiting to see if the put is exercised on you, and if it not within 6 months, using your call. At worst, you have to pay \$600 for the asset (for example, if the put is exercised right away). Your net profit in today's dollars is at least $625 - 600 = 25$. (It could be more, for example, if you purchase the stock in 6 months. But that is not what the question asks.)