Math 441 **Mathematics of Finance** **Spring 2009**

Midterm

March 12, 2009

Name: SOLUTION KEY

Instructions: Show all your work for full credit, and box your answers when appropriate. There are 5 questions. The first is worth 14 points, the second and third are worth 20 points each, and the fourth and fifth are worth 23 points each for a total of 100 points. Unless otherwise noted, all risk free rates are per annum with continuous compounding.

1. Let r_{T_1,T_2} denote the forward rate from (year) T_1 to T_2, . Today is time 0 and today's zero rates are as follows: the 3-month rate is 3%, the 5-month rate is 3.4%, the 8-month rate is 3.9%, the 12-month rate is 3.4%, the 14-month rate is 3.5%. Compute (to one hundredth of one-percent precision) the following rates:

$$r_{\frac{5}{12},\frac{14}{12}} \quad r_{0,1} \quad r_{\frac{5}{12},\frac{9}{12}}.$$

   If not enough information is available for any of these, indicate what additional information you would need to know to compute the rate. Show your work.

   Solution: $r_{\frac{5}{12},\frac{14}{12}}$ satisfies

$$e^{(r_{0,\frac{14}{12}})\times\frac{14}{12}} = e^{(r_{0,\frac{5}{12}})\times\frac{5}{12}} e^{(r_{\frac{5}{12},\frac{14}{12}})\times\frac{14-5}{12}}$$

   which implies $r_{\frac{5}{12},\frac{14}{12}} = 3.55\%$.

   $r_{0,1} = 3.4\%$ is the given 12-month rate.

   $r_{\frac{5}{12},\frac{9}{12}}$ requires knowledge of the 9-month rate.

2. Eighteen months ago, a bank had entered a swap, (original) lifetime of 5 years, agreeing to pay coupons on a fixed bond based on a fixed 5% per annum (annual compounding) and receive coupons on a floating bond based on the one-year LIBOR (annual compounding). Coupons to be exchanged every year. Both bonds have principal (face value) of 1 million dollars. Six months ago, the one-year LIBOR was 5.3% with *continuous* compounding. Today, all rates are 4% with continuous compounding. Compute the value of the swap to the bank.

   Solution: The swap expires in 3.5 years, with the next coupons exchanged in 6 months. The value of the fixed coupon bond is

$$B_{fix} = 50000e^{-0.04*0.5} + 50000e^{-0.04*1.5} + 50000e^{-0.04*2.5} + 1050000e^{-0.04*3.5} = 1054160.$$

To compute the value of the float coupon bond we must first figure out the one-year LIBOR rate six month ago, in terms of annual compounding.

$$(1 + r_{an}) = e^{r_{cts}} = e^{0.053} \quad \rightarrow \quad r_{an} = 0.0544.$$

The value of the float coupon bond is thus

$$B_{fl} = (1000000 + 54400)e^{-0.04*0.5} = 1033550.$$

The value of the swap is

$$V_{swap} = B_{fl} - B_{fix} = -20610.$$

3. The up-front cost of storing potatoes for 6 months is $500 per 10,000 pounds. The spot price of potatoes is $1.10 per pound. The forward price of delivering 10,000 pounds in 6 months is $20,000. Today, you can borrow money (for any length of time) at 5%, and invest money (for any length of time) at 4%. Describe an arbitrage opportunity involving 30,000 pounds of potatoes. What is (in 6 months) the the risk-free profit associated to this opportunity?

   Solution:

   It seems like the forward price is too high relative to the spot price. But let us confirm this with an arbitrage opportunity.

   At time 0, we take a short position promising to deliver 30000 pounds of potatoes in 6 months for $60000. We must buy the potatoes and store them, so we borrow

$$3 * 500 + 30000 * 1.1 = 34500$$

   at 5% for 6 months. With this cash, we buy the potatoes for $33000 and store them paying the up-front cost of $1500.

   In 6 months, we deliver the potatoes for $60000 and repay our loan of $34500e^{0.05*0.5} = 35373$ for a risk-free profit of $24627.

   Note that there is not a well-defined answer for the present value of this profit, since it is unclear if we should discount at 4% or 5% (or something else). The grading reflected this.

4. Consider Bond A which is an 8%-coupon bearing bond maturing in 15 months, with principal $100. Coupons are paid annually. Consider Bond B which is a 0%-coupon bond with principal $100 maturing in one year. Consider Bond C whose current price is $95 and duration is 9 months. Suppose the zero rates are the following: 3 month is 5.0%, 6 month is 5.1%, 9 month is 5.2%, 12 month is 5.3%, 15 month is 5.4%.

(a) What are the prices and durations of Bonds A, B and C?

Solution:

Let P_X and D_X denote the price and duration of bond/portfolio X.

$$\begin{aligned} P_A &= 8e^{-0.05*0.25} + 108e^{-0.54*1.25} = \$108.85 \\ D_A &= \frac{0.25 * 8e^{-0.05*0.25} + 1.25 * 108e^{-0.54*1.25}}{108.85} = 1.17\,\text{years.} \end{aligned}$$

$P_B = 100e^{-0.053} = 94.83$. $D_B = 1$ since there are no coupons. $P_C = 95, D_C = 0.75$ are given.

(b) Your portfolio, which we call E, is made up of the following: a long position in 5 Bond A's; a short position in 2 Bond B's; and a long position in 4 Bond C's. Compute the value (price) and duration of Portfolio E.

Solution:

$$\begin{aligned} P_E &= 5P_A - 2P_B + 4P_C = \$734.51 \\ D_E &= \frac{5P_AD_A - 2P_BD_B + 4P_CD_C}{P_E} = 1.002\text{years.} \end{aligned}$$

(c) Suppose the Federal reserve decreases all rates by 25 bips. Estimate the new value of your portfolio.

Solution:

Let $\Delta r = -0.0025$ be the rate change.

$$P_E^{new} \approx P_E - D_E P_E \Delta r = \$736.42$$

5. The spot price of Google stock is \$300, today March 12, 2009. Google will pay each share holder a \$10/share dividend in four months and a \$10/share dividend in eight months. The 1-month, 2-month, ..., 6-month zero rates are all 3%. The 7-month, 8-month, ..., 12-month zero rates are all 5%.

(a) Compute the forward price of Google stock with delivery date March 12, 2010.

Solution:

$$\begin{aligned} F_{March12} &= S^{eff}_{March12} e^{(\text{rate}*\text{time until delivery})} \\ &= (300 - 10e^{-0.03*4/12} - 10e^{-0.05*8/12})e^{0.05*1} = 294.81. \end{aligned}$$

(b) Suppose 6 months later, on September 12, 2009, the *forward price* of Google stock with delivery date March 12 2010 is now \$295. Assume all rates are 4%. What does that imply the *spot* price of Google should be, assuming no-arbitrage?

Solution:

$$\begin{aligned} F_{Sept12} &= S^{eff}_{Sept12} e^{(\text{rate}*\text{time until delivery})} \quad \text{so} \\ 295 &= (S_{Sept12} - 10e^{-0.04*2/12})e^{0.04*6/12} \end{aligned}$$

so $S_{Sept12} = 299.09$.

(c) Suppose, on March 12, 2009, you had taken a short position on a Google forward with delivery March 12, 2010. What would the value of your position be on September 12, 2009?

Solution: The value of the position is what you will get for the share on 3/12/10, less what anyone else will get for the Google share who enters a forward contract today (9/12/09), discounted by 6 months from 3/12/10 back to 9/12/09. This equals

$$(F_{March12} - F_{Sept12})e^{-(\text{rate}*\text{time until delivery})} = (294.81 - 295)e^{-0.04*0.5} = \$0.186.$$