Math 441 **Mathematics of Finance** **Spring 2013**

Final

May 08, 2013

Name: ____________________________

Instructions: Show all your work for full credit, and box your answers when appropriate. Unless otherwise noted, the risk free rate is per annum with continuous compounding, stocks do not pay dividends, and there are no arbitrage opportunities.

1. Google (GOOG) is currently trading for \$600. The risk-free rate is 3%. A 6-month GOOG **European** put with strike 585 is trading for \$20 more than the 6-month GOOG European call with strike 585. Construct an arbitrage opportunity involving one of each of these options. What is the guaranteed risk-free profit of this strategy in today's dollars?

   Solution: Compare the two sides of put-call parity:

$$\begin{aligned} c + Ke^{-rT} &= c + 585e^{-0.03\times 6/12} = c + 576.29 \\ p + S_0 &= c + 20 + 600 = c + 625 \end{aligned}$$

   The put side is overpriced compared to the call side, so buy call, sell put, sell share. Profit is $625 - 576.29 = 43.71$.

2. You have entered a bet where your payoff is +\$10 if the price of Microsoft (MSFT) increases today (say from when the market opens at 9AM to when it closes at 4:30PM) and −\$10 if MSFT decreases today. Assume that everyday, MSFT has a 60% chance of increasing and a 40% chance of decreasing. Note that the amount MSFT goes up or down is not important here.

   (a) Compute the mean and standard deviation of your bet.

   (b) You decide to make the bet everyday for 30 days. Assume that whether MSFT increases or decreases on any one day is independent of whether it increases of decreases on any other day. Use part (a) and the Central Limit Theorem, to express (an approximation of)your total earnings for the 30 days in the form $a + bZ$ where a and b are constants and Z is a standard normal variable (with mean 0 and standard deviation 1).

   Solution: Let Y_i be the payoff on day i.

$$\begin{aligned} E[Y_i] &= 0.6 \times 10 + 0.4 \times (-10) = 2. \\ SD(Y_i) &= \sqrt{E[Y_i^2] - E[Y_i]^2} = \sqrt{0.6 \times 100 + 0.4 \times 100 - 2^2} = 9.80. \end{aligned}$$

By the CLT, the payoff after 30 days $Y_1 + \ldots + Y_{30}$ is approximately

$$2 \times 30 + 9.8 \times \sqrt{30}Z = 60 + 50.94Z.$$

3. The risk-free rate is 3%. Facebook (FB) currently prices at \$25. The Delta of a 1-year at-the-money European call on FB is 0.5. Mike has a portfolio made of the following:

   - One bond maturing in 1 year paying \$1000.
   - Three 1-year at-the-money European FB calls.
   - Two 1-year at-the-money European FB puts.

   How many shares of FB should Mike long (or short) to make his portfolio delta-neutral? Answer should be to nearest 0.1 shares

   Solution: The Delta of the portfolio Π is

   $$\Delta_\Pi = \frac{\partial \Pi}{\partial S} = \frac{\partial \text{bond}}{\partial S} + 3\frac{\partial \text{call}}{\partial S} + 2\frac{\partial \text{put}}{\partial S} = 0 + 3\Delta_c + 2(\Delta_c - 1) = 0.5.$$

   Since Delta of each share is $\partial S/\partial S = 1$, Mike must short 0.5 shares.

4. The risk-free rate is 3%. The spot price of IBM (IBM) is \$200. Assume IBM stock price follows the lognormal model. The volatility is 20%. Compute the price of a one-year at-the-money European straddle sells.

   Solution: The price of the straddle is

   $$p + c = (c + Ke^{-rT} - S_0) + c = 2c - 5.91$$

   Let us try 20% volatility.

   $$d_1 = \frac{\ln(200/200) + (0.03 + 0.2^2/2)1}{0.2\sqrt{1}} = 0.25, \quad d_2 = \frac{\ln(200/200) + (0.03 - 0.2^2/2)1}{0.2\sqrt{1}} = 0.05.$$

   $$c = 200F_Z(0.25) - 200F_Z(0.05)e^{-0.3} = 200(0.5987) - 200(0.5199)e^{-0.3} = 18.83$$

   So the price is \$31.74.

5. The risk-free rate is 2%. Apple (AAPL) trades today at \$400. Assume that S_t, the price of AAPL S_t, is a solution to the stochastic differential equation

   $$dS = 0.04S_t dt + 0.2S_t dZ.$$

Mike and Hongkun each have \$1000. Mike invests it all in AAPL (fractional shares OK). Hongkun invests it all in a bond maturing in 10 years. What is the probability (to the nearest 1/10 of one percent) that Mike will have at least twice as much money as Hongkun in 10 years?

Solution: Let M_t and H_t denote Mike's and Hongkun's portfolio values at time t.

$$H_t = 1000e^{0.02t}, \quad M_t = 1000/400 S_t = 1000e^{(0.04-\frac{0.2^2}{2})t+0.2Z_t}$$

Write Brownian motion after 10 years as $Z_{10} = \sqrt{10}Z$ for some standard normal variable Z. So we want to compute

$$\begin{aligned}
P(M_{10} \geq 2H_{10}) &= P(1000e^{(0.04-\frac{0.2^2}{2})10+0.2\sqrt{10}Z} > 2000e^{0.02\times 10}) \\
&= P\left((0.04 - \frac{0.2^2}{2})10 + 0.2\sqrt{10}Z > \ln(2) + 0.02 \times 10\right) \\
&= P\left(Z > \frac{\ln(2) + 0.02 \times 10 - (0.04 - \frac{0.2^2}{2})10}{0.2\sqrt{10}}\right) \\
&= P(Z > 1.096) = 1 - F_Z(1.096) = 0.137
\end{aligned}$$

So Mike has a 13.7% chance of having at least twice as much as Hongkun after 10 years.

6. Consider the Weiner process X_t where $dX = 4dt + 3dZ$ and $X_0 = 8$. Consider the stochastic process $Y_t = f(t, X_t)$ where $f(t, x) = t + x^2$.

   (a) Compute dY.

   (b) Use part (a) to estimate $SD(Y_{0.1})$.

Solution: Compute partials $f_t = 1, f_x = 2x, f_{xx} = 2$ we then apply Ito's Lemma to get

$$dY = (1 + 4 \times 2X_t + 3^2/2 \times 2)dt + (2X_t \times 3)dZ.$$

Since (approximately) $Y_{0.1} = Y_0 + dY$ where we plug in $t = 0, dt = 0.1$ into dY we get

$$\begin{aligned}
Y_{0.1} &= Y_0 + (1 + 4 \times 2X_0 + 3^2/2 \times 2)(0.1) + (2X_0 \times 3)(Z_{0.1} - Z_0) \\
&= (0 + 8^2) + (1 + 4 \times 2 \times 8 + 3^2/2 \times 2)(0.1) + (2 \times 8 \times 3)(Z_{0.1} - 0).
\end{aligned}$$

And since $SD(a + bX) = SD(X)$ for constants a, b and r.v. X we get

$$SD(Y_{0.1}) = 2 \times 8 \times 3 \times SD(Z_{0.1}) = 48\sqrt{0.1} = 15.18.$$