Math 441 | **Mathematics of Finance** | **Fall 2013**

Final
December 11, 2013

Name: ______________________________

Instructions: Show all your work for full credit, and box your answers when appropriate. Unless otherwise noted, the risk free rate is per annum with continuous compounding and there are no arbitrage opportunities.

1. The risk-free rate is 3%. Today the spot price of AAPL is \$600. The asset will pay no dividends during the next year. A one-year European put on AAPL with strike \$700 is priced at \$70. Construct an arbitrage opportunity involving one of these puts, and compute the profit from this strategy in today's dollars.

   Solution:
   $$p + S_0 = p + S_0 = 70 + 600 = 670$$
   which is less than
   $$Ke^{-rT} = 700e^{-0.03} = 679.31$$
   So long the put and the asset at $t = 0$, and in 1 year, sell the asset for at least \$700 (either at market price, or at put strike, whichever is greater). You are guaranteed a profit in today's dollars worth at least $679.31 - 670 = 9.31$.

2. Rank the prices of the various positions. If one is not comparable to other, say so. (For example, if you wrote "$A = B > D, A > C$, C and D are incomparable" you indicate how any two positions compare, if they are comparable. You can also write longhand, "$A = B$ $A > C$, $B > C$ etc," but that may take a while.)

   The risk-free rate is 0%. All options below are European with the same underlying asset, and the asset does not pay any dividends over the next year.

   A. A long position in a \$40-50 one year bull spread made of puts.

   B. A long position in a \$40-50 one year bull spread made of calls.

   C. \$10 today and a short position in a \$10-strike one-year put.

   D. A long position in a \$20-30-40 one-year butterfly made of calls.

   Solution: Graphing the payoffs of the positions in one year (and noting that there is no time value of money), $C > B > A$ and $C > D > A$, and B and D are incomparable.

3. The spot price (at $t = 0$) of GOOG is \$1000 and the asset will not pay any dividends over the next year. Every 3 months, GOOG will either go up by 15% or down by 10%. You have just written **ten** 6-month European calls on GOOG with strike 950. Suppose in 3 months GOOG goes down. You decide (at $t = 3/12$) to make your position delta-neutral. How many GOOG shares should you buy or sell?

   Solution: You are at the node where GOOG trades for $1000 \times 0.9 = \$900$ and can either increase to $900 \times 1.15 = \$1035$ or decrease to $900 \times 0.9 = \$810$. In this binomial tree set-up, making your position delta neutral is the same as making it risk-free. The number of shares N needed satisfies the equations (first one is risk-free approach, second one is delta-neutral approach)

   $$1035N + 10 \times (1035 - 95)) = 810N + 10 \times 0 \text{ OR } N = -10\frac{(1035 - 950) - 0}{1035 - 810} = 34/9.$$

4. Today ($t = 0$), the spot price of GOOG is \$1000, the spot price of AAPL is \$600, and neither asset will pay any dividends over the next year. The difference in price is modeled by the stochastic process

   $$X_t = -400 + 150t^2 + 100tZ_t$$

   where Z_t is Brownian motion and t is in years. What is the probability that in two years, the price of AAPL will be \$200 higher than the price of GOOG?

   Solution: Since $X_0 = -400$, X_t represents the price of AAPL$-$GOOG, and not GOOG$-$AAPL.

   $$P(X_2 > 200) = P(-400 + 600 + 200Z_2 > 200) = P(Z_2 > 0/200) = 0.5.$$

5. Let S_t denote the price of FB at time t. The spot price of FB is \$50 and the asset will not pay any dividends over the next year. Suppose in the real world

   $$dS = 0.08S_t dt + 0.16S_t dZ$$

   and in the risk-neutral world

   $$dS = 0S_t dt + 0.16S_t dZ.$$

   Compute the price of derivative whose payoff in one-year is $\Phi(S_1)$ where $\Phi(s) = s^2 + 1000\ln(s)$.

Solution: The price h is the discounted expected payoff in the risk-neutral world.

$$\begin{aligned}
h &= e^{-0\times 1} E_{RWN}[S_1^2 + 1000\ln(S_1)] \\
&= E\left[\left(50e^{(0-0.16^2/2)\times 1+0.16\sqrt{1}Z}\right)^2 + 1000\ln\left(50e^{(0-0.16^2/2)\times 1+0.16\sqrt{1}Z}\right)\right] \\
&= 50^2 e^{-0.16^2} E[e^{0.32Z}] + 1000(\ln(50) - 0.16^2/2 + 0.16E[Z]) \\
&= 50^2 e^{-0.16^2} e^{0.32^2/2} + 1000(\ln(50) - 0.16^2/2) \\
&= 6928.
\end{aligned}$$

6. The spot price of FB is \$50 and the asset will not pay any dividends over the next year. The price satisfies Geometric Brownian motion, but you do not know the FB stock volatility, the expected rate of return, or the risk-free rate of return. However, you do know that the market prices at \$0.25 a one-year FB **digital call** with strike price 60, and that the market prices at \$2 at a one-year FB **European call** with strike price 60. Deduce as much as one can about the **delta** of a one-year FB **American call** with strike price 60.

   Solution: Since the price of the American and European call options are always the same (if no dividends during options' lifetime), so are their deltas, Δ. The formulas for the prices of the digital call h_{dc} and European call h_c are

   $$h_{dc} = e^{-rT} F_Z(d_2), \quad h_c = S_0 F_Z(d_1) - K F_Z(d_2) e^{-rT} = 50 F_Z(d_1) - 60 h_{dc}.$$

   The delta of the European (and hence American) call is thus

   $$\Delta = F_Z(d_1) = \frac{h_c + 60 h_{dc}}{50} = \frac{2 + 60 \times 0.25}{50} = 0.34.$$