3.8 Wave Equation

Modeling waves and more specifically dispersal of particles in bodies of water inside canals or rivers is of great interest and has many applications worldwide. More specifically, the particles in question may range from simple organic pollutants such wash off from farm lands to chemical or nuclear waste products which have been illegally damped in the water. The benefits of such knowledge can be environmental as well as economical and at times such knowledge may be lifesaving. Modeling the flow of particles in a river through its usual land path is actually not so difficult since essentially the flow is mostly constant. It is only when the river intersects with other rivers or when it reaches the end of its journey that the flow of particles change. When rivers or canals are reaching their end which may be a lake or the ocean there are other effects present. The flow is more erratic and is normally non-constant in these situations. Under these circumstances we may observe changes in both the flow as well as the direction of the flow of the particles we are tracking.

Other common applications of the wave equation involve modeling the vibrations of a string. It is with respect to this application in fact that we will derive the wave equation itself.

3.8.1 Derivation for the wave equation

We will pursue the study of a vibrating string here by first simplifying our problem into a roughly one-dimensional domain. This is not entirely unrealistic based on the fact that a string has a small cross section and can be considered almost as thin as a line. So this simplification is not too severe. We will study this string under the influence of its internal forces only. So no gravity or other external forces (friction) will be taken into account for this derivation. Note that you can easily implement any outside forces by adding the appropriate extra term in this derivation. See exercise for a case where you are require to include the effects of gravity in this model.

Let us first introduce out notation for the variables to be studied here:

- $u(x,t)$ is the displacement of the string with respect to x and t.
- $\theta(x,t)$ the angle between the string and the horizontal at x and t.
- $T(x,t)$ the tension on the string at location x and time t.
- $\rho(x)$ the density of the string.

As usual the main idea behind any derivation using mechanics is to look at the forces acting on a body. We will start by applying Newton's second law which states that mass times acceleration equals all forces acting on a body,

$$\text{mass} \cdot \text{acceleration} = \text{sum of all forces}$$

Let us start by finding out the relevant expressions for each of the quantities above.

It is unfortunately not possible here to include graphics but please refer to the picture drawn on the board in class which depicts a small section of the string which we will study. Note in that picture how we separate all forces acting on the string into their vertical and horizontal components.

- *Vertical direction:* We consider, as usual, a small section of the string with length Δx and density $\rho(x)$. Then, as we have seen in other derivations before, the mass of that section of the string is

$$m = \text{Density} \cdot \text{Length} = \rho(x)\sqrt{\Delta x^2 + \Delta u^2}. \tag{3.30}$$

We also know that any body whose position is $u(x,t)$ has acceleration $\partial^2 u/\partial t^2$. Thus the force due to Newton in the vertical direction is

$$ma = \rho(x)\sqrt{\Delta x^2 + \Delta u^2}\frac{\partial^2 u}{\partial t^2} \tag{3.31}$$

Putting together all forces in the vertical direction acting on our string therefore gives,

$$\rho(x)\sqrt{\Delta x^2 + \Delta u^2}\frac{\partial^2 u}{\partial t^2} = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t).$$

This for instance would be where you could add, as an extra term, the effects of other *vertical* acting forces such as gravity. Dividing this equation by Δx we obtain,

$$\rho(x)\sqrt{1+\frac{\Delta u^2}{\Delta x^2}}\frac{\partial^2 u}{\partial t^2} = \frac{T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t)}{\Delta x}.$$

Note that the right hand side of the above is actually a finite difference of $T\sin\theta$ with respect to x. In that respect all terms involving Δ will become actual derivatives if we let $\Delta x \to 0$. Applying $\Delta x \to 0$ on the above we obtain something that resembles a differential equation,

$$\rho(x)\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\big(T(x,t)\sin\theta(x,t)\big)$$

Therefore from the product rule the right hand side of the above becomes,

$$\rho(x)\sqrt{1+\left(\frac{\partial u}{\partial x}\right)^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial T(x,t)}{\partial x}\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\frac{\partial \theta(x,t)}{\partial x}$$

Note that this equation has too many variables and therefore is not very useful as is. We therefore will try to simplify it when possible.

Is is true that

$$\tan\theta(x,t) = \frac{\partial u(x,t)}{\partial x} \quad \text{and therefore also that:} \quad \theta(x,t) = \tan^{-1}\left(\frac{\partial u(x,t)}{\partial x}\right)$$

Taking derivative of $theta(x,t)$ above gives,

$$\frac{\partial \theta(x,t)}{\partial x} = \frac{\frac{\partial^2 u(x,t)}{\partial x^2}}{\sqrt{1+(\frac{\partial u(x,t)}{\partial x})^2}} \quad \cos\theta(x,t)$$

Similarly the following relations hold:

$$\sin\theta(x,t) = \frac{\frac{\partial u(x,t)}{\partial x}}{\sqrt{1+(\frac{\partial u(x,t)}{\partial x})^2}} \quad \cos\theta(x,t) = \frac{1}{\sqrt{1+(\frac{\partial u(x,t)}{\partial x})^2}}$$

These formulas can all be used to eliminate all the θ's from our main equation. The result is correct but quite cumbersome to work with. Instead we will make some further simplifying assumptions which can reduce our model drastically.

Main assumption: Assume that $\theta(x,t)$ is very very small (i.e $\theta(x,t) \approx 0$). Then $\tan\theta(x,t) \approx \theta(x,t)$ and therefore also $\frac{\partial u(x,t)}{\partial x} \approx 0$. Thus a number of terms in our previous relations will simplify as follows:

$$\sqrt{1+\left(\frac{\partial u}{\partial t}\right)^2} \approx 1, \quad \sin\theta(x,t) \approx \frac{\partial u(x,t)}{\partial t}, \quad \cos\theta(x,t) \approx 1, \quad \frac{\partial \theta(x,t)}{\partial t} \approx \frac{\partial^2 u(x,t)}{\partial t^2}$$

Therefore our main equation becomes

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial T(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial x} + T(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} \tag{3.32}$$

Still notice that we have one last problem: two unknowns T and u and one equation! Remember however that we have not yet put into use the equivalent formulation from the horizontal direction...

- *Horizontal direction* The forces in the horizontal direction are equal to zero since no movement is taking place. As a result,

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0.$$

Once again dividing by Δx and allowing $\Delta x \to 0$ we obtain a differential form of this equation,

$$\frac{\partial}{\partial x}(T(x,t)\cos\theta(x,t)) = 0 \tag{3.33}$$

Since, as we mentioned earlier $\cos\theta \approx 1$ for small θ then the above implies that $T(x,t)$ must be a constant in with respect to x. Thus $T(x,t)$ is a function of t only!

Therefore based on our finding in (3.33), we can revise our equation in (3.32) as follows,

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T(x,t)\frac{\partial^2 u(x,t)}{\partial x^2}$$

Assuming now that both density ρ and tension T are constants we can obtain our *wave equation*,

$$\frac{\partial^2 u}{\partial t^2} = \nu\frac{\partial^2 u(x,t)}{\partial x^2} \tag{3.34}$$

where $\nu = T(x,t)/\rho(x)$. Note that the assumption that the density $\rho(x)$ is a constant and does not depend on x is realistic with respect to most common materials. Similarly the tension T can also be assumed to be a constant in the case of guitar strings since no-one does really change the tension of the guitar while playing. As a result *nu* is usually a given constant.

3.8.2 Exact solution of wave equation

This time we can solve the equation not just numerically but also exactly. The methods of solutions may vary from separation of variables to method of characteristics. Either way the solution to the wave equation is known to be the sum of two "traveling waves",

$$c(x,t) = C_1\cos(kx-\theta t) + C_2\sin(kx+\theta t) \tag{3.35}$$

where θ denotes the frequency of oscillation and $2\pi/k$ is the "wave length". In fact k and θ relate to ν through the following relationship

$$\nu = \frac{\theta}{k}.$$

The wave amplitudes C_1 and C_2 can be found from given initial and boundary conditions. You can easily check whether (3.35) does solve our wave equation (3.34) by a simple substitution.

This solution in other words displays two waves, one traveling to the left with speed ν and another one traveling to the right with speed ν also.