

A simple example follows:

Example:

Suppose a high frequency transmission line whose length is $L = 121$ meters. We wish to discover how exactly is temperature distributed along the wire. This is important since if at any point the wire temperature is high then transmission will be slowed. Suppose that the initial temperature on the wire is everywhere the same $u(x, 0) = 100$ Celcius and that at each end the temperature is kept at zero degrees with some cooling equipment. The material used in the construction of the wire has a known thermal conductivity constant which is $\nu = 2$. Find the temperature of the wire $u(x, t)$ at any location x in the future.

Solution

This problem can be translated mathematically to a heat equation with *Dirichlet* type conditions. We know that the heat equation is generally considered to be a good model to use if we are to describe the temperature distribution in a rod (or a wire). The term Dirichlet boundary conditions is customarily used when we are given u on the boundary - in contrast we could have *Newmann* boundary conditions which implies that the derivatives of u are known on the boundary instead. Thus our model has the following form,

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = 2 \frac{\partial^2 u}{\partial x^2} & 0 < x < 121, t > 0 \\ u(0, t) = 0 & t > 0 \\ u(121, t) = 0 & t > 0 \\ u(x, 0) = 100 & 0 < x < 121 \end{array} \right.$$

We start as usual by assuming the solution to be $u(x, t) = X(x)T(t)$. We then apply this separation of variables and as seen before we obtain the following two ODEs

$$\begin{array}{l} T'(t) + 2\lambda T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \end{array} \quad \text{with } X(0) = 0 \quad \text{and} \quad X(121) = 0$$

How did the boundary conditions come about? We simply translated the boundary conditions from the original problem. For instance since $u(0, t) = 0$ then it must be true that $X(0)T(t) = 0$. This implies that either $X(0) = 0$ or that $T(t) = 0$. However if we allow $T(t) = 0$ then the whole solution must be zero. We already know that $u(x, t) = 0$ is a possible solution and we are now looking for other. Thus we take the other possibility of $X(0) = 0$. A similar argument will also produce $X(121) = 0$. Note however that we can not use the non-homogeneous condition $u(x, 0) = 100$ yet. This will come in handy in the very end.

We start by solving the 1st order ODE first. The “integrating factor” is $\mu = e^{\int 2\lambda dt} = e^{2\lambda t}$ (please review the “intro to ODEs” part of our notes from the very beginning of the class if you do not see where this came from). As a result the equation becomes

$$(T(t)e^{2\lambda t})' = 0$$

or after integration,

$$T(t) = Ce^{-2\lambda t} \tag{3.25}$$

We now turn to the second ODE. The solution of this ODE should follow one of the cases shown earlier in (3.24). Which one is appropriate is revealed by solving the equivalent quadratic equation

$$w^2 + \lambda w = 0$$

which gives that $w = \pm\sqrt{-\lambda}$. Note that there are three possibilities for λ . Either λ is positive, negative or zero. We explore all three cases below although, as our discussion will show, it is only the case of $\lambda > 0$ which is physically relevant since, in the absence of any external heating, temperature tends to decrease. Note that for $\lambda > 0$ in equation (3.25) the temperature will decrease in time.

- The case of $\lambda < 0$. Thus if $\lambda < 0$ then the roots w_1 and w_2 are both real and non-equal. Therefore the solution, following our chart from the previous page, is

$$X(x) = C_1 \exp(\sqrt{-\lambda}x) + C_2 \exp(-\sqrt{-\lambda}x) \quad (3.26)$$

We can then obtain the values of the constants C_1 and C_2 based on the Dirichlet boundary conditions

$$X(0) = X(121) = 0.$$

which were originally provided for us. Note that substituting $X(0) = 0$ into (3.26) we immediately obtain that

$$C_1 + C_2 = 0. \quad (3.27)$$

Applying now the second boundary condition $X(121) = 0$ into (3.26) gives

$$C_1 \exp(\sqrt{-121\lambda}) + C_2 \exp(-\sqrt{-121\lambda}) = 0. \quad (3.28)$$

So we manage to obtain two equations (3.27) and (3.28) in two unknowns C_1 and C_2 . Solving these two equations for C_1 and C_2 will produce the complete solution (3.26) for $X(x)$. However, as was pointed out earlier this case of $\lambda < 0$ is not physically relevant.

- The case of $\lambda = 0$. If on the other hand $\lambda = 0$ then our original problem $X''(x) + \lambda X(x) = 0$ is reduced to simply

$$X''(x) = 0.$$

This has a very easy solution $X(x) = C_1x + C_2$. Applying the initial conditions $X(0) = X(121) = 0$ we obtain that $X(x) = 0$. This essentially implies that $u(x, t) = X(x)T(t) = 0$ or the so called “trivial solution”. This solution although mathematically possible is also not physical. Therefore we will next explore the last case which, based on our earlier discussion is also making sense based on the physics of the problem.

- The case of $\lambda > 0$. We continue here in order to find other more general solutions. Thus we then examine the last and most important case: $\lambda > 0$. In this case the solution based on (3.24) has the form,

$$X(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$$

Once again the constants C_1 and C_2 can be found from the given conditions. Clearly for $X(0) = 0$ we obtain that $C_1 = 0$. Thus our solution above reduces to,

$$X(x) = C_2 \sin(\sqrt{\lambda}x) \quad (3.29)$$

Note that applying the second condition $X(121) = 0$, and assuming that $C_2 \neq 0$, we can solve for λ . In that case we get

$$0 = \sin(121\sqrt{\lambda}) \quad \text{which gives that} \quad \lambda = \left(\frac{n\pi}{121}\right)^2, \quad \text{for } n = 0, 1, 2, \dots$$

We substitute this value of λ into (3.29). Note that in this case we have a different solution X for each n . We can therefore write these solutions,

$$X_n(x) = C_n \sin\left(\frac{n\pi}{121}x\right) \quad \text{for } n = 0, 1, 2, \dots$$

Since all of the above are solutions then, using the *principle of superposition* we can write all such solutions in a compact form by summing them up,

$$X(x) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi}{121}x\right)$$

Let us now put all pieces of our analysis together and write down the complete solution for our PDE,

$$\begin{array}{l} \text{PDE} \\ \text{Solutions} \end{array} \quad u(x, t) = \begin{cases} [C_1 \exp(\sqrt{\lambda}x) + C_2 \exp(-\sqrt{\lambda}x)] e^{-2\lambda t} & \text{if } \lambda < 0 \\ \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi}{121}x\right) e^{-2\frac{n^2\pi^2}{121^2}t} & \text{if } \lambda > 0 \\ 0 & \text{-the trivial solution-} \quad \text{if } \lambda = 0 \end{cases}$$

It is usually the solution for $\lambda > 0$ which is more natural

$$u(x, t) = \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi}{121}x\right) e^{-2\frac{n^2\pi^2}{121^2}t}$$

since heat is expected to decrease as time goes by (in the absence of any external device adding heat). Still, note that our solution is not complete since the constants C_n have not been found yet. Notice however that we still have not imposed the initial condition that $u(x, 0) = 100$. This condition allows us to completely solve for the constants C_n . This is a very easy task however with the help of *Fourier theory*:

Below we present only the part of Fourier theory which will be useful for our course. We apply the condition $u(x, 0) = 100$. Thus we have to solve

$$100 = \sum_{n=0}^{\infty} C_n \sin\left(\frac{n\pi}{121}x\right) \tag{3.30}$$

for each C_n . This in fact is a so called Fourier series. As a result there exist tools which allow us to calculate the coefficients C_n . This is actually an easy task with the help of Fourier theory.

Theorem 16. *In general for any sine Fourier series of the function $f(x)$ defined on the interval $0 < x < L$,*

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where the unknown coefficients b_n can be found by,

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Based on the results of this theorem the coefficients C_n of (3.30) are defined to be

$$C_n = \frac{2}{121} \int_0^{121} 100 \sin\left(\frac{n\pi}{121}x\right) dx = 200 \frac{1 - \cos(n\pi)}{n\pi} = \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \quad (3.31)$$

This therefore gives the complete solution for our PDE to be,

$$u(x, t) = 200 \sum_{n=0}^{\infty} \frac{1 - \cos((2n+1)\pi)}{(2n+1)\pi} \sin\left(\frac{(2n+1)\pi}{121}x\right) e^{-2\frac{(2n+1)^2\pi^2}{121^2}t}$$

assuming that $\lambda > 0$. Note that we wrote this solution for $2n+1$ instead of n in order to make sure that only the odd values are used.