3.7.3 Numerical Solutions of Heat Equation

As we have seen before we can apply our finite difference method in order to produce an approximate, numerical solution for our heat equation. We follow exactly the same procedure as the one used to solve our linear advection equation. We start, in other words, by discretizing our space time domain

$$\begin{aligned} x = x_m &= mh \quad \text{where } m \text{ denotes the number of space points} \\ t = t_n &= nk \quad \text{and } n \text{ denotes the number of time points} \end{aligned}$$

and replacing our continuous derivatives with discrete equivalent versions. As we have seen before we have several options to choose from by which we can replace our first derivative in time and second derivative in space. For instance,

$$\begin{aligned} &\text{Forward in } t: & u_t(x,t) &= \tfrac{U_{m,n+1}-U_{m,n}}{k} & -O(k) \\ &\text{Centered in } x: & u_{xx}(x,t) &= \tfrac{U_{m+1,n}-2U_{m,n}+U_{m-1,n}}{h^2} & +O(h^2) \\ &\text{Centered in } t: & u_t(x,t) &= \tfrac{U_{m,n+1}-U_{m,n-1}}{2k} & -O(k) \end{aligned}$$

etc... Here as usual

$$U_{m,n} = u(x,t), \quad U_{m+1,n} = u(x+h,k), \quad U_{m-1,n} = u(x-h,k), \quad U_{m,n+1} = u(x,t+k), \quad \text{etc...}$$

Our particular example therefore,

$$u_t = \nu u_{xx}$$

choosing explicit centered finite differences transforms to,

$$\text{Explicit Centered:} \qquad \frac{U_{m,n+1}-U_{m,n}}{k} = \nu \frac{U_{m+1,n}-2U_{m,n}+U_{m-1,n}}{h^2}$$

In order to produce a solution from this numerical scheme we must first rewrite it. We solve for the term(s) which contain the advanced time. For our example there is only one such term $U_{m,n+1}$. Thus we obtain,

$$U_{m,n+1} = sU_{m-1,n} + (1-2s)U_{m,n} + sU_{m+1,n} \tag{3.29}$$

where $s = \nu k/h^2$. Note that the "stencil" has the following shape for space and time nodes,

	n		$n+1$
m+1	$U_{m+1,n}$	$s \searrow$	
m	$U_{m,n}$	$1-2s \rightarrow$	$U_{m,n+1}$
m-1	$U_{m-1,n}$	$s \nearrow$	

Example:
Suppose the following wave equation is provided,

$$\begin{cases} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 4, \quad t > 0 \\ u(0,t) &= 0 \quad t > 0 \\ u(4,t) &= 0 \quad t > 0 \\ u(x,0) &= 5 \quad 0 < x < 4 \end{cases}$$

Produce the solution at $t = 2$ using a time step size of 1 sec. and a spatial step size of 1.

Solution Based on the above $k = 1$ and $h = 1$. Let us now try and produce the solution at time $t = 2$ sec. and any spatial location. We start by calculating s,

$$s = \nu \frac{k}{h^2} = 1\frac{1}{1^2} = 1$$

We first translate our initial and bondary conditions into the U's,

$$\begin{aligned} u(x,0) = 5 &\ \rightarrow U(0,0) = U(1,0) = U(2,0) = U(3,0) = U(4,0) = 5 \\ u(0,t) = 0 &\ \rightarrow U(0,0) = U(0,1) = U(0,2) = 0 \\ U(4,t) = 0 &\ \rightarrow U(4,0) = U(4,1) = U(4,2) = 0 \end{aligned}$$

Thus our complete domain so far looks like this,

$x = 4$	$U_{4,0}$	$U_{4,1}$	$U_{4,2}$
$x = 3$	$U_{3,0}$	$U_{3,1}$	$U_{3,2}$
$x = 2$	$U_{2,0}$	$U_{2,1}$	$U_{2,2}$
$x = 1$	$U_{1,0}$	$U_{1,1}$	$U_{1,2}$
$x = 0$	$U_{0,0}$	$U_{0,1}$	$U_{0,2}$
	$t = 0$	$t = 1$	$t = 2$

substituting the initial and boundary conditions $\longrightarrow$

$x = 4$	5	0	0
$x = 3$	5	?	?
$x = 2$	5	?	?
$x = 1$	5	?	?
$x = 0$	5	0	0
	$t = 0$	$t = 1$	$t = 2$

We must now calculate the solution in the interior of this domain. To do this we apply the values from the table above (initial and boundary conditions) in our numerical scheme (3.29). If you are doing this by hand you may find it easier however to just use the stencil provided in the previous page.

Performing one iteration after another advancing in time and using the appropriate boundary and initial conditions but also previous solutions our scheme (3.17) produces slowly the full solution inside our domain as follows:

5	0	0
5	?	?
5	?	?
5	?	?
5	0	0

$\xrightarrow{t=1}$

5	0	0
5	5	?
5	5	?
5	5	?
5	0	0

$\xrightarrow{t=2}$

5	0	0
5	5	0
5	5	5
5	5	0
5	0	0

Therefore the numerical solution is given by the last column above $u^T(x, 2) = [0\ 0\ 5\ 0\ 0]$. This solution is supposed to be only an approximation to the true solution at time $t = 2$ for our heat equation .