

## 3.7 Heat Equation

We will now look at a model for describing the distribution of temperature in a solid material as a function of time and space. It is probably easy to see the need for such a description and applications of it to our everyday life are abound. As usual there are several simplifications and assumptions behind such a model. Among them is the assumption that the body, which we wish to measure the temperature of, is homogeneous (i.e. it is composed of the exact same material and no foreign bodies are in it). There do exist more complicated models however which can describe the temperature distribution in non-homogeneous bodies. Similarly we will assume that our object is perfectly insulated from surrounding sources (or sinks) of heat and in fact that it can not generate heat of each own.

To simplify the mathematical modeling aspect further we will first attempt to solve this problem in just one spatial dimension. So we consider:

- a (homogeneous material) rod of length  $L$
- a constant cross section  $S$
- further assume that the rod is completely insulated everywhere around.

The well known mathematical model (which we will derive below) that makes use of these assumptions and describes the temperature distribution in three dimensions is the famous heat (or diffusion) equation,

$$u_t = \nu (u_{xx} + u_{yy} + u_{zz}) \quad (3.18)$$

where  $\nu$  is known *thermal diffusivity* and is related to, as well as calculated from, properties of the material. The heat equation is in fact another realization of the conservation of energy which we saw a version of in the Euler equations. We present below the derivation of this model

### 3.7.1 Derivation of the heat equation

There are several approaches in order to obtain a mathematical model for temperature through a body. Mainly these approaches are characterized into two main categories:

- Relativistic
- Newtonian

Although much more interesting mathematically the relativistic approach, which produces the exact same equation (3.18), is too complicated to present here in detail. This method relies on atomistic properties and generates a system of equations which produce (3.18). We will instead present the classical Newton approach. In this case we represent our variables of interest through classical conservation laws which we know that any model relating them ought to adhere to. Thus in a similar fashion as we did for the derivation of our linear advection equation we now consider an infinitesimal element in our rod and write the conservation of energy law for it.

We first provide some necessary background from physics. It is well-known from the *Fourier heat conduction law* that heat  $Q$  is transported in direction opposite to the temperature gradient of  $u$  and is proportional to it,

$$Q(x, t) = -\kappa \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z} \right) \quad (3.19)$$

where  $\kappa$  is the proportionality constant also known as *thermal conductivity*. It is also known that heat  $Q$  is related to mass  $m$  and temperature  $u$  via the following formula,

$$Q(x, t) = \lambda mu(x, t) \quad (3.20)$$

where  $\lambda$  is the known *specific heat* for our material.

Let us complete this derivation for now in just one dimension  $x$ . Note however that this result can be very easily generalized to all three dimensions. Consider an infinitesimal piece from our rod with length  $[x, x + \Delta x]$ . Then if the rod has cross section  $S$  this piece has volume  $S\Delta x$ . Further assuming that the density of the material for the body under consideration is  $c$  then the infinitesimal mass for our infinitesimal volume element is simply given by

$$\Delta m = cS\Delta x.$$

Therefore based on (3.20) the equivalent heat for our volume element is described by,

$$Q = \lambda mu = \lambda cS\Delta xu \quad (3.21)$$

Let us now take into account physical properties. In other words that the energy, or heat, in any piece of the insulated rod ought to be conserved. Considering again our infinitesimal volume piece with length  $[x, x + \Delta x]$  we can state the following,

rate of change of energy (heat) = rate heat flowing in - rate heat flowing out

or mathematically written as,

$$\frac{\partial Q}{\partial t} = Q_{in}(x, t)S - Q_{out}(x + \Delta x, t)S \quad (3.22)$$

Note however that the rate of change of energy (heat) can be found by differentiating our equation in (3.21),

$$\frac{\partial Q}{\partial t} = \lambda cS\Delta x \frac{\partial u}{\partial t} \quad (3.23)$$

We substitute this for the left hand side of (3.22) and obtain,

$$\lambda cS\Delta x \frac{\partial u}{\partial t} = S(Q(x, t) - Q(x + \Delta x, t))$$

Let us rewrite the above by dividing with  $\Delta x$  and  $S$ ,

$$\nu c \frac{\partial u}{\partial t} = - \frac{Q(x + \Delta x, t) - Q(x, t)}{\Delta x}$$

Note however that the right hand side of the above is simply the derivative of  $Q$  with respect to  $x$ ,  $\partial Q/\partial x$  if we let  $\Delta x \rightarrow \infty$ . Thus we obtain,

$$\lambda c \frac{\partial u}{\partial t} = - \frac{\partial Q}{\partial x}$$

Finally using the Fourier law of heat conduction (3.19) in one dimension,  $Q = -\kappa du/dx$ , we obtain our heat equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

where  $\nu = \kappa/(\lambda c)$  groups all parameters together. This derivation can be easily generalized to higher dimensions.

### 3.7.2 Solutions of the Heat equation

There are several different methods which allow knowledge of exact or numerical solutions for the heat equation. Before however we even start discussing the possible solutions of this equation we should get acquainted with some amazing properties that these solutions tend to have. To shed light into this subject we use the well known maximum principle which in fact can be applied to more general problems with similar success. We give below a version of the maximum principle as applied to the heat equation.

**Theorem 15.** *Suppose the heat equation (3.21) is defined on the following time - space domain,  $0 < t < T$  and  $a \leq x \leq b$ . Then the temperature  $u$  attains both its maximum and minimum values on the boundary of this domain! Further that temperature is unique.*

#### Theoretical solutions - separation of variables method

Let us examine a concrete example of a rod length  $L$  with temperature  $u(x, t)$ . We keep the ends of the rod  $x = 0$  and at  $x = L$  at a constant temperature  $u = 0$  while we will set the initial temperature in the rod to be increasing with distance via  $u(x, 0) = x$ . In other words we propose the following heat equation model for this example,

$$\left\{ \begin{array}{ll} \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < L, \quad t > 0 \\ u(0, t) = 0 & \text{for } t > 0 \\ u(L, t) = 0 & \text{for } t > 0 \\ u(x, 0) = x & 0 < x < L \end{array} \right.$$

We will apply a very general procedure called separation of variables in order to obtain the solution to this model. The idea behind this procedure is general enough that can be successfully applied to several other problems with similar results. The separation of variables method assumes that the solution  $u(x, t)$  which we are looking for can be separated into a product of solutions each of which depends only on one variable. In other words we assume

$$u(x, t) = X(x)T(t)$$

We then obtain the time and 2nd space derivative of the above,

$$u_t = X(x)T'(t) \quad \text{and} \quad u_{xx} = X''(x)T(t)$$

Substituting into our heat equation above we obtain the following,

$$X(x)T'(t) = \nu X''(x)T(t)$$

Let us now separate variables by moving all terms with a  $t$  to the left and all the terms with an  $x$  to the right,

$$\frac{T'(t)}{\nu T(t)} = \frac{X''(x)}{X(x)}$$

It is important to note here that if we pick a specific time  $t = t_1$  then our equation above becomes

$$\frac{T'(t_1)}{\nu T(t_1)} = \text{constant} = \frac{X''(x)}{X(x)}$$

for all  $x$ ! Similarly if we pick a specific space value  $x = x_1$  then it becomes,

$$\frac{T'(t)}{\nu T(t)} = \frac{X''(x_1)}{X(x_1)} = \text{constant}$$

for all  $x$ ! All in all the following information has been revealed,

$$\frac{T'(t)}{\nu T(t)} = \text{constant} \quad \text{and} \quad \frac{X''(x)}{X(x)} = \text{constant}$$

Note that we have successfully reduced the original PDE problem into two very simple ODE problems. We will solve both of the above ODEs and produce the values of  $T(t)$  and  $X(x)$ . Once this is done then we can reassemble the solution of our original PDE from the fact that  $u(x, t) = X(x)T(t)$ .

It is customary to denote the general constant above by  $-\lambda$  and we therefore have the following two ODEs to solve,

$$\frac{T'(t)}{\nu T(t)} = -\lambda \quad \text{and} \quad \frac{X''(x)}{X(x)} = -\lambda$$

which we can rewrite as,

$$\begin{aligned} T'(t) + \lambda\nu T(t) &= 0 \\ X''(x) + \lambda X(x) &= 0 \end{aligned}$$

It is easy to obtain the solutions for each of those equations. In general any linear second order homogeneous equation such as  $X''(x) + aX'(x) + bX(x) = 0$  is first transformed to a quadratic formula

$$w^2 + aw + b = 0.$$

We solve it using the quadratic formula. Depending on the two solutions of this quadratic  $w_1$  and  $w_2$  we have the following solutions for the ODE,

$$\begin{array}{l} \text{General} \\ \text{2nd order} \\ \text{ODE Solutions} \end{array} \left\{ \begin{array}{ll} X(x) = C_1 \exp(w_1 x) + C_2 \exp(w_2 x) & \text{if } w_1 \neq w_2 \text{ and both are real} \\ X(x) = C_1 \exp(w_1 x) + C_2 x \exp(w_2 x) & \text{if } w_1 = w_2 \text{ and both are real} \\ X(x) = \exp(\alpha x)(C_1 \cos(\beta x) + C_2 \sin(\beta x)) & \text{if } w \text{ is complex: } w = \alpha + i\beta \end{array} \right. \quad (3.24)$$

Similarly we obtain the solutions for our first order ODE  $T'(t) + \lambda\nu T(t) = 0$  with one of the methods which we have already learned previously. If for instance we use the separation of variables method for this ODE we obtain the solution to be

$$T(t) = T(t_0)e^{-\lambda\nu(t-t_0)}$$

As a result the total solution to our heat equation model is just the product of both the first order ODE and the second order ODE together  $u(x, t) = X(x)T(t)$ .