3.6.1 Numerical Solutions by Finite Differences

We will now explore a numerical method which will allow us to produce approximations to the solutions of the models which we have presented so far. This will be the well known finite difference method. This method is capable of easily, with the help of computers, producing solutions to a vast number of ODEs and PDEs.

The main idea behind the finite difference method is that we can approximate successfully the derivatives in a given differential equation by their first order Taylor approximations as follows:

$$
\begin{array}{llll}
\text{Forward in } t: & u_t(x,t) & = \frac{u(x,t+k)-u(x,t)}{k} & -O(k) \\
\text{Forward in } x: & u_x(x,t) & = \frac{u(x+h,t)-u(x,t)}{h} & +O(h) \\
\text{Centered in } x: & u_{xx}(x,t) & = \frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2} & +O(h^2) \\
\text{Centered in } x: & u_x(x,t) & = \frac{u(x+h,t)-u(x-h,t)}{2h} & +O(h^2) \\
\text{Centered in } t: & u_t(x,t) & = \frac{u(x,t+k)-u(x,t-k)}{2k} & +O(k^2)
\end{array}
\tag{3.14}
$$

where k denotes the change in time (Δt) and h the change in space (Δx). Let us however see exactly how this method, these so called divided differences, can be applied in a specific model in order to produce its solution. We will start with a general linear advection equation but for the non-homogeneous case,

$$u_t + au_x = f \tag{3.15}$$

Here we will assume as usual that a is a constant. Also notice that now we include a non-homogeneous piece f which we will allow to depend on both space and time, $f = f(x,t)$.

Let us start by discretizing the space-time domain into m and n points respectively. Thus any point in space $x = x_m$ and time $t = t_n$ will now be denoted by

$$
\begin{aligned}
x_m &= mh \\
t_n &= nk
\end{aligned}
$$

Before we go any further we should simplify the notation used in (3.14) above and replace it with the following simpler version,

$$
\begin{array}{lll}
u_t(x,t) & = \frac{U_{m,n+1}-U_{m,n}}{k} & -O(k) \\
u_x(x,t) & = \frac{U_{m+1,n}-U_{m,n}}{h} & +O(h) \\
u_{xx}(x,t) & = \frac{U_{m+1,n}-2U_{m,n}+U_{m-1,n}}{h^2} & +O(h^2)
\end{array}
$$

where as you can see here the following notation has been used:

$$U_{m,n} = u(x,t), \quad U_{m+1,n} = u(x+h,k), \quad U_{m-1,n} = u(x-h,k), \quad U_{m,n+1} = u(x,t+k), \quad \text{etc...}$$

Based on these ideas we are now in position to replace our exact linear advection model equation (3.15) with an approximate version which we should be able to solve. We introduce below several choices of numerical schemes all of which can be applied in order to produce the solution of our linear advection equation $u_t + au_x = f$

$$
\begin{array}{lr}
\text{Explicit Forward:} & \frac{U_{m,n+1}-U_{m,n}}{k} + a\frac{U_{m+1,n}-U_{m,n}}{h} = f_{m,n} \\
\text{Explicit Backward:} & \frac{U_{m,n+1}-U_{m,n}}{k} + a\frac{U_{m,n}-U_{m-1,n}}{h} = f_{m,n} \\
\text{Implicit Forward:} & \frac{U_{m+1,n+1}-U_{m+1,n}}{k} + a\frac{U_{m+1,n+1}-U_{m,n+1}}{h} = f_{m,n} \\
\text{Centered time \& Centered space (Leapfrog):} & \frac{U_{m,n+1}-U_{m,n-1}}{2k} + a\frac{U_{m+1,n}-U_{m-1,n}}{2h} = f_{m,n}
\end{array}
$$

Do you see where they come from? Do you understand how their names come about? Could you make your own formulas which would correspond to our model (3.15)? These are typically the main tasks that you must become familiar with.

Let us now use one of these schemes and try to obtain the solution for our model equation $u_t + au_x = f$. Let's choose for example the following explicit center space scheme,

$$\frac{U_{m,n+1} - U_{m,n}}{k} + a\frac{U_{m+1,n} - U_{m-1,n}}{2h} = f_{m,n}$$

In order to produce a solution from this numerical scheme we must first rewrite it. We solve for the term(s) which contain the advanced time. For our example there is only one such term $U_{m,n+1}$. Thus we obtain,

$$U_{m,n+1} = U_{m,n} - s(U_{m+1,n} - U_{m-1,n}) + kf_{m,n}$$

where $s = ak/2h$. This can be simplified to,

$$U_{m,n+1} = sU_{m+1,n} + (1 - s)U_{m,n} + sU_{m-1,n} + kf_{m,n}$$

Using this scheme let us now produce the approximate numerical solution for a specific PDE.
Example: Solve numerically the following advection equation with the given Cauchy type conditions. Produce the approximate solution at $t = 3$ with a time step size of $k = 1$ and a space step size of $h = .5$ where $L = 1.5$.

$$\begin{cases} u_t + 2u_x &= 0, \quad \text{for all } 0 \le x \le L, \quad t \ge 0 \\ u(x,0) &= 1, \quad \text{for all } 0 \le x \le L \\ u(0,t) &= 0, \quad \text{for all } t > 0 \\ u(L,t) &= 1, \quad \text{for all } t > 0 \end{cases} \tag{3.16}$$

Solution: We want to produce the solution at $t = 3$ provided a step side of $k = 1$ and a step size for space $h = .5$ with $L = 1.5$.

Given this set-up and since $f = 0, a = 2, h = .5$ and $k = 1$ then $s = 2$. Our numerical scheme above simplifies to the following,

$$U_{m,n+1} = 2U_{m+1,n} - U_{m,n} + 2U_{m-1,n} \tag{3.17}$$

Let us also discretize the given initial conditions so that we can apply them to our numerical scheme above. The equation $u(x,0) = 1$ simply implies that initially $U_{0,0} = U_{1,0} = U_{2,0} = U_{3,0} = 1$. Similarly the boundary conditions $u(0,t) = 0$ and $u(2,t) = 1$ imply that we know the following values $U_{0,1} = U_{0,2} = U_{0,3} = 0$ and $U_{3,0} = U_{3,1} = U_{3,2} = U_{3,3} = 1$ respectively. In other words our total domain so far (at time $t = 0$) looks like this:

x				
	$U_{3,0}$	$U_{3,1}$	$U_{3,2}$	$U_{3,3}$
	$U_{2,0}$	$U_{2,1}$	$U_{2,2}$	$U_{2,3}$
	$U_{1,0}$	$U_{1,1}$	$U_{1,2}$	$U_{1,3}$
	$U_{0,0}$	$U_{0,1}$	$U_{0,2}$	$U_{0,3}$
	t			

$\xrightarrow{t=0}$

x				
	1	1	1	1
	1	?	?	?
	1	?	?	?
	1	0	0	0

We must find out the solution in the interior of this domain. To do this we apply the given initial and boundary conditions in our numerical scheme (3.17).

Performing one iteration after another advancing in time and using the appropriate boundary and initial conditions but also previous solutions our scheme (3.17) produces slowly the full solution inside our domain as follows:

1	1	1	1
1	?	?	?
1	?	?	?
1	0	0	0

$\xrightarrow{t=1}$ x

1	1	1	1
1	**3**	?	?
1	**3**	?	?
1	0	0	0

$\xrightarrow{t=2}$ x

1	1	1	1
1	3	**5**	?
1	3	**3**	?
1	0	0	0

$\xrightarrow{t=3}$ x

1	1	1	1
1	3	5	**3**
1	3	3	**7**
1	0	0	0

Therefore the numerical solution at final time $t = 3$ is $u(0,3) = 0, u(.5,3) = 7, u(1,3) = 3$ and $u(1.5,3) = 1$. Note that this solution is supposed to be only an approximation to the true solution of our linear advection model (3.16). Naturally if you take smaller time steps k and space steps h this approximation can become a lot better.