3.6 Burger's Equation

One of the most famous equations in applied mathematics is Burger's equation which originally was derived in order to model fluid dynamics and even turbulence although not with much success. Among the many reasons for the popularity of this equation is that it seems to "spring up" as a simplification of many far more difficult models. So most researchers will usually try their ideas on the simpler model, which in many instances turns out to be Burger's equation, and then using what they have learned try to obtain solutions and properties for their original but more general model. The equation is described as follows,

$$u_t + uu_x = 0 \tag{3.13}$$

Seems somewhat familiar? Well in fact note that it is equivalent to our linear advection equation $u_t + q(u)_x = 0$ for the special case of

$$q(u) = \frac{1}{2}u^2$$

Go ahead, try this short calculation to persuade yourself that they are the same!

Burger's equation although seemingly simple may have a very rich spectrum of possible solutions. Thus is can easily transform from a very simple easy to solve equation to one that is very rich and could conceivably produce realistic behavior for the physical process it describes. As such Burger's equation seems a perfect tool in order to get acquainted with the intricacies of hyperbolic equations and their solutions.

It may be interesting to note that in fact equation (3.13) is not the real Burger's equation but only a so called "inviscid" (non-viscus) version of it. The true Burger's equation is a bit more diffusive,

$$u_t + uu_x = \nu u_{xx} \qquad \text{- Burger's eq. with viscosity}$$

since it includes this extra second derivative term u_{xx} which accounts for viscosity (or otherwise called diffusion). Here ν denotes the viscosity constant.

Let us consider initial conditions for Burger's equations which are a bit different than those we used before. Before we considered the Cauchy problem type of initial conditions which essentially amounted to some known smooth wave to be provided at time $t = 0$. Now however we will consider the Riemann problem which has the following set-up,

$$\text{Riemann problem:} \qquad u(x,0) = \begin{cases} u_L(x) & \text{for all } x < 0 \\ u_R(x) & \text{for all } t \geq 0 \end{cases}$$

where here u_L and u_R are some given, usually constant, values of the wave in the left and right side of $x = 0$ at $t = 0$. In other words the Riemann problem is not smooth and in fact contains an initial condition which includes a discontinuity at $x = 0$. This usually makes the problem harder to solve.

Some of the interesting features of Burger's equation is that its solutions contain several of the important features seen in fluids such as *shocks* and *rarefactions*. Rarefaction is the term we use for waves that are smooth and in general not ready to break. In contrast shocks appear in waves when they are steep and essentially breaking. Usually shocks and rarefactions appear as solutions of Burger's equation when we solve it together with this Riemann type of initial conditions and is attributed to the discontinuity at $x = 0$.

The Reynolds number is one of many useful measures we have in order to understand and mostly categorize general aspects of the behavior of fluids and is defined as,

$$\text{Reynolds } \# = \frac{\text{Inertia Force}}{\text{Viscous Force}}$$

So clearly based on this definition if the viscous forces are small or similarly ν = small then the Reynolds number is large and visa versa. For small Reynolds numbers we observe the rarefied wave phenomenon while for high Reynolds numbers shocks appear.

Last it can be shown that using the Cole-Hopf transformation

$$u = -2\nu \frac{\phi_x}{\phi}$$

Burger's equation can be reduced to another well known equation, the heat equation

$$\phi_t = \nu \phi_{xx}$$