3.9 Modeling and Finance

We now take a quick look into the modeling of finance but also more generally stochastic differential equations. In this Chapter we will try to give an overview of some of the basic mathematical methods and models which have been used to predict trends for stock and more generally get an edge and make money over other investors.

Before we begin we must provide some definitions and basic theorems which will be useful in obtaining the solutions and other results from our financial models. There are several books which you can follow if you would like a more in depth view on the subject of finance. In our approach we will follow closely the examples in Chapters 4, 5 and 12 from "Stochastic Differential Equations" by Bernt Oksendal, 5th edition. We encourage the interested student to look into this book as well as several other references therein for all the details.

3.10 Background and basic definitions

One of the most well known examples of how mathematical models can be successful in predicting the value of a stock is the well known *Black and Scholes formula*. The Black and Scholes formula is nothing more than a stochastic differential equation. Fischer Black and Myron Scholes were the first to successfully come up with such an equation in order to predict how much money an investor should pay for a so called "European Stock Option". What is a European Stock Option? It is the option to buy in the future (if you still want to) a given stock at a pre-specified price. First of all you do not have to buy the stock in the future if you do not want to. You just pay an agreed upon amount now so that you can have this option later. You agree in advance however, when you can exercise your option to buy (not before or after) and how much you must pay for the stock at that time. Needless to say that if the value of the stock has risen a lot you will be making a ton of money if you have agreed to buy the stock at a smaller value. Naturally in the opposite situation you may not want to buy the stock at all and just incur only the loss of the initial amount for buying the option in the first place. Options, nowadays, is a big business. Even governments get into the action of dealing in options. You can therefore imagine how useful it would be to have a tool which could predict the future value of a given stock! In 1973 Black and Scholes produced their successful formula capable of giving such a prediction. For their discovery they were awarded the Nobel price in economics in 1997.

The mathematics necessary to understand and solve such mathematical models of finance are extremely complicated and require several years of in depth study in measure theory, analysis, differential equations etc. In this Chapter we will only look at a limited few models and methods of solution. We hope that this introduction to the mathematical treatment of financial models will interest the reader in a further course in the mathematics of finance.

Let us start with some basic definitions. We define $X_i(t)$ to be the price of stock i at a given time t. A *market* $X(t)$ is the collection of all stocks

$$X(t) = (X_0(t), X_1(t), \dots, X_n(t)).$$

Usually an investor will try to diversify their overall position by buying a certain amount of a given stock and not placing all her/his holdings into just a single stock. We call the overall position or holdings that an investor has placed for different stocks a *portfolio*. We denote by $\theta_i(t)$ the number of units of stock i that the investor holds at time t. Therefore the portfolio for that investor in a market of $n+1$ stocks is denoted by

$$\theta(t) = (\theta_0(t), \theta_1(t), \ldots, \theta_n(t))$$

The overall *value of portfolio* $\theta(t)$ for our investor at any given time is therefore found from,

$$V(t) = \theta(t) \cdot X(t) = \theta_0(t)X_0(t) + \theta_1(t)X_0(t) + \cdots + \theta_n(t)X_n(t)$$

In order to produce the solutions to any of our models in finance we must learn the basics about solutions of stochastic differential equations. This is done in the next section.

3.11 Stochastic Differential Equations

A stochastic differential equation is nothing more than a deterministic differential equation (derivatives of the function) plus "noise". What is noise? Noise incorporates random effects into our deterministic differential equation. Since nature is not deterministic is makes sense to want to model natural behavior with models which do include this noise (or random) effects.

Let us look at a simple example. We revisit our simple population model from Chapter 1 in our book,

$$\frac{dN}{dt} = a(t)N(t)$$

where $N(t)$ denotes the population size at time t and $a(t)$ is the corresponding rate of growth at that time. If $a(t)$ is known completely then the above differential equation is deterministic and easy to solve with the methods which we have already examined in class. If however the rate $a(t)$ is not deterministic but instead it also contains random terms,

$$a(t) = r(t) + \text{"}noise\text{"}$$

then we have a stochastic differential equation,

$$\frac{dN}{dt} = (r(t) + \text{"}noise\text{"})N(t)$$

where $r(t)$ is completely known and "noise" represents the random effects. Although the behavior of this noise term is not known we do assume that its probability distribution is known and is in fact the same as that of a Gaussian distribution. What is a Gaussian distribution? A simple example of a Gaussian distribution is that of the probability distribution corresponding to throwing a die. A die can take the values from 1 to 6 and has an average of 3.5. This is a typical example of a Gaussian distribution. In other words if you repeated this experiment thousands of times and plotted the histogram you would obtain a bell shaped curve which has its high pick for the value of 3.5 and is symmetric around that value.

Let us now look into an outline of how we would solve such a stochastic differential equation. Although in general it is not easy to solve differential equations, it is even harder to solve stochastic differential equations.

The main method of solution behind both stochastic or regular differential equations is actually the same. The main difference in the solution is knowing how to take an integral of a stochastic process such as $N(t)$. In fact this is the only new item we must learn before we are capable of solving stochastic differential equations. This new method of integration of processes which include stochastic terms is called *Ito Calculus*. Before we present however this new method of integration let us look at a complete example of a solution of a stochastic differential equation (SDE).

3.11.1 An Example

In this subsection we present an example of how we normally solve an SDE (Stochastic Differential Equation) in hope of motivating at least the outline of the method. Although not all steps in the solution will be clear please try to follow it so that you will become familiar with the general procedure anyway. The missing steps (the Ito formula) will be taken care in the next section.

The best way to present the solution of a SDE is to instead produce the solution of a usual differential equation. The reason for looking at a differential equation instead is that the solution methods are in fact the same. In that respect we will start with a deterministic differential equation which we have seen before: the population model

$$\frac{dN(t)}{dt} = aN(t)$$

where a is a constant (no random effects yet). Also note that for now at least this is just a simple, deterministic, ODE. The method of solution for such an equation is actually very simple. We will just apply the separation of variables method and obtain the solution $N(t)$.

We therefore start by simply separating variables,

$$\frac{dN(t)}{N(t)} = adt$$

and integrating both sides. We will use definite integrals and assume that at initial time, $t = 0$, the population is known to be $N(0)$. We want to find the population at a given final time $t = s$. Thus integrating the above we have,

$$\int_0^s \frac{dN(t)}{N(t)} = \int_0^s adt.$$

The integrals above are both easy,

$$\ln N(t) \ \Big]_0^s = at \ \Big]_0^s$$

Substituting in the values for t and simplifying the algebra we obtain the solution for the population at time $t = s$ to be,

$$N(s) = N(0)e^{as}$$

Let us now look at the solution of a very similar population model. This time however we will include the random effects in the growth rate a of the population. Let us therefore assume that the growth rate depends on both a deterministic rate r which is a known constant and a "noisy" random term, which will be denoted by $W(t)$,

$$a(t) = r + W(t).$$

Thus our differential equation now is a SDE,

$$\frac{dN(t)}{dt} = (r + W(t))N(t)$$

We rewrite the above as,

$$\frac{dN(t)}{dt} = rN(t) + W(t)N(t)$$

Note that now the unknown population $N(t)$ is what we call a stochastic process and it will in fact be the reason for our troubles when we will try to integrate it later. For now we follow the same procedure at solving this SDE regardless. Thus we first separate variables and then integrate both sides,

$$\frac{dN(t)}{N(t)} = rdt + W(t)dt.$$

We will go a step further and use the notation

$$dB(t) = W(t)dt$$

where $B(t)$ is what we call "Brownian motion". Thus our main SDE can be written us,

$$\frac{dN(t)}{N(t)} = rdt + dB(t).$$

Let us integrate both sides as before,

$$\int_0^s \frac{dN(t)}{N(t)} = \int_0^s rdt + \int_0^s dB(t)$$

We start with our right hand side. First,

$$\int_0^s rdt = rt \ \Big]_0^s = rs$$

Similarly,

$$\int_0^s dB(t) = B(t) \ \Big]_0^s = B(s) - B(0) = B(s).$$

So far this does not look so difficult after all. But now we will attempt to integrate the left hand side and that is where the trouble starts. Our usual calculus is in fact wrong when applied to a stochastic process such as $N(t)$. We must use stochastic calculus rules for such processes. In this class we will in fact learn some of the basics of Ito calculus and how to use it in order to correctly obtain the answers from such integrals. In order to integrate the term above we need to make use of something called the "Ito formula". Although we learn about the Ito formula in the next section let us give the answer here even though it will not make sense where it came from. For our integral

$$\int_0^s \frac{dN(t)}{N(t)}$$

we know that the function which we would normally obtain using usual calculus and replacing N by x is in fact,

$$g(t, x) = \ln x$$

The Ito formula (which you do not know yet) for this function $g(t, x)$ with $x = N(t)$ gives,

$$d(\ln N(t)) = \frac{dN(t)}{N(t)} - \frac{1}{2}dt \tag{3.36}$$

You are not supposed to understand where the above formula came from. This you will learn in the next section. So for now just accept this and see if you can follow the remaining of this solution.

Let us now rewrite (3.36) as,

$$\frac{dN(t)}{N(t)} = d(\ln N(t)) + \frac{1}{2}dt$$

and integrate it,

$$\int_0^s \frac{dN(t)}{N(t)} = \int_0^s d(\ln N(t)) + \int_0^s \frac{1}{2}dt \tag{3.37}$$

The right hand side of the above gives,

$$\ln N(t) \ \Big]_0^s + \frac{1}{2}t \ \Big]_0^s = \ln N(s) - \ln N(0) + \frac{1}{2}s = \ln\left(\frac{N(s)}{N(0)}\right) + \frac{1}{2}s$$

Thus substituting this into (3.37) we now have the answer for our previously unknown stochastic integral,

$$\int_0^s \frac{dN(t)}{N(t)} = \ln\left(\frac{N(s)}{N(0)}\right) + \frac{1}{2}s$$

The above was obtained making use of the Ito formula and is in fact what we call Ito integration. Notice that in fact the Ito integral is similar in some way to the usual integral plus a correction term $-\frac{1}{2}s$. It is possible for some cases that there will be no correction term. But in fact we should follow this procedure anyway just to make sure.

Let us put together the complete solution of this SDE,

$$\ln\left(\frac{N(s)}{N(0)}\right) + \frac{1}{2}s = rs + B(s)$$

and solving for $N(s)$,

$$N(s) = N(0)e^{rs - \frac{1}{2}s + B(s)}$$

Note that the solutions between the ODE and SDE are similar in form (they are both exponentials) but slightly different. In fact the biggest difference is that the SDE solution contains the Brownian motion term $B(s)$. It is generally believed that the solution of the SDE model can capture more realistic terms than the ODE model solution can. Nevertheless if the model is not good to start with it does not matter whether it has random effects incorporated in it or not. Thus we should always keep in mind that the underlying model formulation is more important than anything else and it should always be our most important concern. Once we are satisfied with the model itself we can start considering whether some parameters in the model would be better to be represented via stochastic terms...

Going through the solution of the SDE we had to make use of the Ito formula which gave rise to a new type of integral for stochastic processes. How did that formula come about? We present below a quick overview, without proofs, of Ito calculus and in particular *Ito integration.*