3.13 SDE Models and Examples

In this section we put into use all of the new skills which were introduced up to now. We will apply Ito calculus and more particularly the Ito formula and stochastic integration into solving SDE models. The method of solution in general follows the exact same steps as in usual ODEs. The examples which we present below will come from applications in many different settings and sometimes in multi-dimensions. We start our exposition however with the simpler cases.

Example 1: Consider the motion of a particle through a fluid after being initially prepared in a certain state of motion (e.g., after being pushed by an external force). As a result of friction with the fluid, the particle will be slowed down (i.e., its initial kinetic energy will be dissipated by heating up the medium). The motion of such particle is described by the generalized Langevin equation, which is derived in this section as follows. Solve the *Langevin* equation (or *Ornstein-Uhlenbeck* equation) with constant coefficients,

$$dX(t) = \mu X(t)dt + \sigma dB(t)$$

where μ, σ are real constants and $B(t) \in \mathbb{R}$ is as usual the 1-dimensional Brownian motion. The solution $X(t)$ is called the Ornstein-Uhlenbeck process.

To solve this equation we must multiply with the appropriate integrating factor in exactly the same way as we would for a usual ODE. For this linear type SDE the integrating factor is $\exp(-\mu t)$. Thus we multiply our SDE with $\exp(-\mu t)$,

$$e^{-\mu t}dX(t) = e^{-\mu t}\mu X(t)dt + \sigma e^{-\mu t}dB(t)$$

Let us rewrite the above in a slightly easier form to work with,

$$e^{-\mu t}dX(t) - e^{-\mu t}\mu X(t)dt = \sigma e^{-\mu t}dB(t)$$

We are now ready to integrate the above:

$$\int_0^s e^{-\mu t}dX(t) - \int_0^s e^{-\mu t}\mu X(t)dt = \int_0^s \sigma e^{-\mu t}dB(t) \tag{3.41}$$

Let us look at each of the integrals above. First let's take a look on the left hand side of this equation. Under usual calculus the integral and the derivative would cancel out on the left hand side and we would obtain the following,

$$e^{-\mu t}X(t)$$

Therefore, for the integrals on the left hand side, in order to use Ito integration, we will choose our function g to be,

$$Y(t) = g(t, x) = e^{-\mu t}x$$

where $x = X(t)$. Now we are ready to apply the Ito formula for g. The Ito formula gives,

$$dY(t) = -\mu e^{-\mu t}X(t)dt + e^{-\mu t}dX(t)$$

Replacing $Y(t)$ above we obtain,

$$d(X(t)e^{-\mu t}) = -\mu e^{-\mu t}X(t)dt + e^{-\mu t}dX(t)$$

Integrating this we finally obtain the answer to our unknown integral in (3.41),

$$\int_0^s e^{-\mu t} dX(t) - \int_0^s e^{-\mu t} \mu X(t) dt = \int_0^s d(X(t) e^{-\mu t}).$$

Thus, based on this results, (3.41) becomes,

$$\int_0^s d(X(t) e^{-\mu t}) = \int_0^s \sigma e^{-\mu t} dB(t)$$

which gives,

$$X(s) e^{-\mu s} - X(0) = \int_0^s \sigma e^{-\mu t} dB(t)$$

or better written,

$$X(s) = X(0) e^{\mu s} + \sigma \int_0^s e^{-\mu(t-s)} dB(t)$$

where σ is a constant and has been taken out of the integral. You can try to finish the integration on the right hand side by another application of Ito formula and an appropriate function $g(t,x)$. We leave this as an exercise to the interested student. If you do try this you will notice that you will get a solution but not a complete solution...

Example 2: The charge $Q(t)$ at time t at a fixed point in an electric circuit satisfies the differential equation

$$LQ''(t) + RQ'(t) + \frac{1}{C} Q(t) = F(t) \quad \text{with } Q(0) = Q_0, Q'(0) = I_0 \text{ given}$$

where L is inductance, R is resistance, C is capacitance and $F(t)$ the potential source at a given time t. All these parameters L, R, C are known constants and similarly the source function $F(t)$ is assumed known. The problem specified so far is completely deterministic and can be solved with our ordinary methods for ODE and usual calculus. However certain models of electric circuits assume that the source (or some of the other coefficients) are not deterministic but instead stochastic of the following form,

$$F(t) = G(t) + \text{"}noise\text{"} = G(t) + \alpha W(t)$$

where as usual $W(t)$ denotes the Wiener "white noise" and α is just a constant. We wish to solve such a model for an electric circuit and obtain a representation of the charge $Q(t)$ at any time t.

Let us start by using some of our more familiar notation. At the same time we must also introduce vector notation since this is a second order SDE and as such we must reduce it first into two first order SDEs. This practice is common to us from our studies in ODEs in the beginning of the semester. We therefore introduce the vector,

$$\bar{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} Q(t) \\ Q'(t) \end{pmatrix}$$

which reduces our second order SDE above into the following 2 SDEs both of which are of first order,

$$\begin{aligned} X_1' &= X_2 \\ LX_2' &= -RX_2 - \tfrac{1}{C} X_1 + G(t) + \alpha W(t) \end{aligned}$$

Note that this now is a two dimensional SDE. It is going to be easier to represent the above in shorthand notation as follows

$$d\bar{X}(t) = A\bar{X}(t)dt + H(t)dt + KdB(t) \tag{3.42}$$

where $\bar{X}(t), H(t)$ and K are vectors and A is a matrix as follows,

$$d\bar{X}(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} \quad H(t) = \begin{pmatrix} 0 \\ \frac{1}{L}G(t) \end{pmatrix} \quad K(t) = \begin{pmatrix} 0 \\ \frac{\alpha}{L} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}$$

The above equation can be written in shorter form as,

$$d\bar{X}(t) = A\bar{X}(t)dt + \frac{1}{L}F(t)dt \tag{3.43}$$

where $F(t)dt = (G(t) + \alpha W(t))dt$. Note that (3.43) is a linear type differential equation and as such we must first find the integrating factor μ and then multiply (3.43) by it. Since in this case $\mu = \exp(-At)$ we obtain from (3.43) the following equation,

$$\mu d\bar{X}(t) - \mu A\bar{X}(t)dt = \frac{\mu}{L}F(t)\, dt$$

The above therefore becomes,

$$e^{-At}d\bar{X}(t) - e^{-At}A\bar{X}(t)dt = \frac{e^{-At}}{L}F(t)\, dt$$

We must now integrate both sides above.

$$\int_0^s e^{-At}d\bar{X}(t) - \int_0^s e^{-At}A\bar{X}(t)dt = \frac{1}{L}\int_0^s e^{-At}F(t)\, dt \tag{3.44}$$

As usual we are not concerned as much for the integral of the right hand side. It is instead the integral on the left hand side which is tougher and for which we need to use Ito calculus since $\bar{X}(t)$ is a stochastic process. We must therefore apply the equivalent 2-dimensional Ito formula for the following 2-coordinate functions g_1, g_2 of

$$g : [0,\infty) \times R^2 \to R^2 \qquad \text{given by} \qquad \begin{pmatrix} g_1(t,x_1,x_2) \\ g_2(t,x_1,x_2) \end{pmatrix} = \exp(-At)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The two-dimensional Ito formula for this function g therefore gives,

$$d(e^{-At}\bar{X}(t)) = (-A)e^{-At}\bar{X}(t)dt + e^{-At}d\bar{X}(t).$$

Integrating this Ito formula we get,

$$\int_0^s (-A)e^{-At}\bar{X}(t)dt + e^{-At}d\bar{X}(t) = \int_0^s d(e^{-At}\bar{X}(t))$$

whose left hand side is exactly the integral we were trying to evaluate in (3.44). Thus note that based on this result (3.44) becomes,

$$\int_0^s d(e^{-At}\bar{X}(t)) = \frac{1}{L}\int_0^s e^{-At}F(t)\, dt$$

Integrating the left hand side we obtain the solution,

$$e^{-As}\bar{X}(s) - \bar{X}(0) = \frac{1}{L}\int_0^s e^{-At}F(t)\ dt$$

or

$$\bar{X}(s) = e^{As}\bar{X}(0) + \frac{e^{As}}{L}\int_0^s e^{-At}F(t)\ dt \tag{3.45}$$

Since $F(t)dt = (G(t) + \alpha W(t))dt = G(t)dt + \alpha B(t) = H(t)dt + KdB(t)$ (from (3.42)) the integral on the right hand side above can be writen as,

$$\begin{aligned} \frac{e^{As}}{L}\int_0^s e^{-At}F(t)\ dt &= e^{As}\int_0^s e^{-At}H(t)\ dt + e^{As}\int_0^s e^{-At}K\ dB(t) \\ &= KB(t) + e^{As}\int_0^s e^{-At}[H(t) + AK]\ dt \end{aligned} \tag{3.46}$$

where the last equality is attributed to using integration by parts for stochastic integrals. Thus putting together all the pieces (3.46) in (3.45) we have the complete simplified solution,

$$\bar{X}(s) = e^{As}\bar{X}(0) + KB(t) + e^{As}\int_0^s e^{-At}[H(t) + AK]\ dt$$

So the charge $Q(s)$ for the circuit at time $t = s$ is given by the top component of the above solution (remember that to start with we set $Q(s) = X_1(s)$. On the other hand we also know the current $I(s)$ since that is the bottom component of this equation. Thus,

$$\begin{aligned} Q(s) = X_1(s) &= e^{As}Q(0) + e^{As}\textstyle\int_0^s e^{-At}\frac{\alpha}{L}\ dt = \ldots \text{can be simplified further...} \\ I(s) = X_2(s) &= e^{As}I(0) + \frac{\alpha}{L}B(t) + e^{As}\textstyle\int_0^s e^{-At}[\frac{1}{L}G(t) - \frac{\alpha R}{L^2}]\ dt. \end{aligned}$$